ON THE MEASURE DIVISION CONSTRUCTION OF Λ COALESCENTS

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Abstract. This paper provides a new construction of Λ coalescents called "measure division construction". This construction consists in dividing the characteristic measure Λ into several parts and adding them one by one to have a whole process. Using this method, we investigate the time back to the MRCA (most recent common ancestor) of a Bolthausen-Sznitman coalescent perturbed by a signed measure. We also study the branch lengths of some other coalescent processes "around" Boltausen-Sznitman coalescent.

1. Introduction

1.1. **Motivation and main results.** Let $\mathbb{N} := \{1, 2, \cdots\}$, Ω be a subset of $\mathbb{N}$ and π a partition of Ω such that $|\pi| < +\infty$ ($|\pi|$ denotes the number of blocks in π). The Λ coalescent process starting from π, introduced independently by Pitman[24] and Sagitov[25], is denoted by $\Pi^{(\Lambda,\pi)} := (\Pi^{(\Lambda,\pi)}(t))_{t\geq 0}$, where $\Pi^{(\Lambda,\pi)}(0) = \pi$ and Λ is a finite measure on $[0, 1]$. Here we specify that a finite measure on $[0, 1]$ can be a null measure and hence its total mass is a non-negative real value. If $\pi = \{\{1\}, \{2\}, \cdots, \{n\}\}$, i.e. the set of first n singletons, then the process is simply denoted by $\Pi^{(\Lambda,n)}$. For any finite measure Λ' on $[0, 1]$, we call the Λ coalescent process with $\Lambda = \Lambda'$ "Λ' coalescent". We assume that all the measures throughout this paper are supported in $[0, 1]$.

This process $\Pi^{(\Lambda,\pi)}$ is a continuous time Markov chain with càdlàg trajectories taking values in the set of partitions of Ω. More precisely: Assume that at time t, $\Pi^{(\Lambda,\pi)}(t)$ has b blocks, then after a random exponential time with parameter

$$g_b^{(\Lambda)} := \sum_{k=2}^{b} \binom{b}{k} \int_0^1 x^{k-2}(1-x)^{b-k}\Lambda(dx), \tag{1}$$

$\Pi^{(\Lambda,\pi)}$ encounters a collision and the probability for a group of $k (2 \leq k \leq b)$ blocks to be merged with the other $b - k$ blocks unchanged is

$$\frac{\int_0^1 x^{k-2}(1-x)^{b-k}\Lambda(dx)}{g_b^{(\Lambda)}}.$$

Then

$$p_{b,b-k+1}^{(\Lambda)} := \frac{\binom{b}{k}\int_0^1 x^{k-2}(1-x)^{b-k}\Lambda(dx)}{g_b^{(\Lambda)}}, \tag{2}$$

is the probability to have $b - k + 1$ blocks after the collision.

Remark that if $\Lambda(\{0\}) = 0$, then we get the following well known formula:

$$g_b^{(\Lambda)} = \int_0^1 (1-(1-x)^b - bx(1-x)^{b-1})x^{-2}\Lambda(dx). \tag{3}$$

The definition shows that the law of $\Pi^{(\Lambda,\pi)}$ is determined by the initial value π and the measure Λ which is hence called characteristic measure.

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Notice that Ω can be an abstract set and the coalescing mechanism works all the same. The reason why one takes Ω as a subset of $\mathbb{N}$ relies on its applications in the genealogies of populations. We take $\Pi^{(\Lambda,n)}$ as an example where $\Omega=\{1,2,\cdots,n\}$. At time 0, we have $\Pi^{(\Lambda,n)}(0)=\{\{1\},\{2\},\cdots,\{n\}\}$ which is interpreted as a sample of n individuals labelled from 1 to n . If at time t, $\Pi^{(\Lambda,n)}$ has its first coalescence where $\{1\}$ and $\{2\}$ are merged together with the others unchanged, then $\Pi^{(\Lambda,n)}(t)=\{\{1,2\},\{3\},\cdots,\{n\}\}$ which is interpreted as getting the MRCA (most recent common ancestor) $\{1,2\}$ of individuals 1 and 2 with the others unchanged at that time, etc. Hence $\{1,2,\cdots,n\}$ is an absorption state of $\Pi^{(\Lambda,n)}$ and is the MRCA of all individuals. For more details, we refer to [20, 22] or [1, 6, 13, 18].

The definition shows that right before a collision, all blocks are exchangeable to be selected for the collision. This property is called exchangeability.

Let $1\leq m\leq n$ and σ the restriction from $\{1,2,\cdots,n\}$ to $\{1,2,\cdots,m\}$. We have the consistency property: σ o $\Pi^{(\Lambda,n)}\overset{(d)}{=}\Pi^{(\Lambda,m)}$ (see [24]). According to this property, if π' is a subset of π, then the restriction of $\Pi^{(\Lambda,\pi)}$ from π to π' has the same distribution as that of $\Pi^{(\Lambda,\pi')}$. Notice that the restriction is from path to path. We can define $\Pi^{(\Lambda,\pi)}$ when $|\pi|=+\infty$ by using the consistency property and the definition in finite case (see [24]).

Let $|\Pi^{(\Lambda,n)}|$ be the block counting process associated to $\Pi^{(\Lambda,n)}$. Then it decreases from n at time 0. We denote by $\tau^{(\Lambda,n)}$the time back to the MRCA of $\Pi^{(\Lambda,n)}$ and by $X_1^{(\Lambda,n)}$ the decrease of number of blocks at the first coalescence. For $i\in\{1,\dots,n\}$, we define

$$T_i^{(\Lambda,n)}:=\inf\left\{t\geq 0|\,\{i\}\notin\Pi_t^{(\Lambda,n)}\right\}$$

the length of the ith external branch and $T^{(\Lambda,n)}$ the length of a randomly chosen external branch. By exchangeability, $T_i^{(\Lambda,n)}\overset{(d)}{=}T^{(\Lambda,n)}$. We denote by $L_{ext}^{(\Lambda,n)}:=\sum_{i=1}^n T_i^{(\Lambda,n)}$ the total external branch length of $\Pi^{(\Lambda,n)}$, and by $L^{(\Lambda,n)}$ the total branch length.

There are four classes of Λ coalescents having been largely studied. We give some results concerning one external branch length $T^{(\Lambda,n)}$. It shows a common regularity that we will discuss later.

- $\Lambda=\delta_0$: Kingman coalescent (see [20], [21]). Then $nT^{(\Lambda,n)}$ is asymptotically distributed with density function $\frac{8}{(2+x)^3}\mathbf{1}_{x\geq 0}$ (See [4], [7], [19]).
- $\Lambda=\Lambda^{leb}$: Bolthausen-Sznitman coalescent (see [5]). Here Λ^{leb} means the lebesgue measure on $[0,1]$. Then $\ln nT^{(\Lambda,n)}$ converges in distribution to $Exp(1)$ (we denote by $Exp(r), r>0$, the exponential variable with parameter r)[15].
- $\Lambda(dx)/dx=\frac{x^{a-1}(1-x)^{b-1}}{Beta(a,b)}\mathbf{1}_{0\leq x\leq 1}, 0<a<1, b>0$: $Beta(a,b)$ coalescent. Here $Beta(\cdot,\cdot)$ denotes Euler's beta function. Then $n^{1-a}T^{(\Lambda,n)}$ converges in distribution to a random variable $T(a,b)$ which has density function $\frac{\Gamma(a+b)}{(1-a)\Gamma(b)}(1+\frac{\Gamma(a+b)}{(2-a)\Gamma(b)}x)^{-\frac{3-2a}{1-a}}\mathbf{1}_{x\geq 0}$ (see [10]).
- $\int_0^1 x^{-1}\Lambda(dx)<+\infty$: These processes are called coalescents without proper frequencies. This category contains $Beta(a,b)$ coalescents with $1<a<2, b>0$ (see [24], [26]). Then $\left(\int_0^1 x^{-1}\Lambda(dx)\right)T^{(\Lambda,n)}$ converges in distribution to $Exp(1)$ (see [17], [23]).

We see a common property for the last three cases concerning one external branch length which is that the normalization factor for $T^{(\Lambda,n)}$ is $\mu^{(\Lambda,n)}:=\int_{1/n}^1 x^{-1}\Lambda(dx)$. More precisely,

- Bolthausen-Sznitman coalescent: Notice that $\mu^{(\Lambda,n)}=\ln n$. Hence directly we have $\mu^{(\Lambda,n)}T^{(\Lambda,n)}\overset{(d)}{\to}Exp(1)$.
- $Beta(a,b)$ coalescent with $0<a<,b>0$:

$$\mu^{(\Lambda,n)}=\int_{1/n}^1 x^{-1}\Lambda(dx)=\int_{1/n}^1\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}x^{a-2}(1-x)^{b-1}dx=\frac{\Gamma(a+b)}{(1-a)\Gamma(a)\Gamma(b)}n^{1-a}+O(1).$$

  Hence $\mu^{(\Lambda,n)}T^{(\Lambda,n)}$ converges in distribution to $T(a,b)\Gamma(a+b)/(1-a)\Gamma(a)\Gamma(b)$.

- If $\int_0^1 x^{-1}\Lambda(dx) < +\infty$, then $\lim_{n\to+\infty} \frac{\mu^{(\Lambda,n)}}{\int_0^1 x^{-1}\Lambda(dx)} = 1$. Hence $\mu^{(\Lambda,n)}T^{(\Lambda,n)}$ converges in distribution to $Exp(1)$.

Kingman coalescent can be viewed as the informal limit of $Beta(a,b)$ coalescent with $0 < a < 1, b > 0$ when a tends to 0, since the measure $\frac{x^{a-1}(1-x)^{b-1}}{Beta(a,b)}\mathbf{1}_{0\leq x\leq 1}$ tends weakly to the Dirac measure on 0. The normalization factor in the case of $Beta(a,b)$ coalescent is n^{1-a}, and of Kingman coalescent is n. Hence we see that these two factors show also some kind of continuity as a tends to 0. We can informally take n as $\mu^{(\Lambda,n)}$ in the case of Kingman coalescent.

Therefore $\mu^{(\Lambda,n)}$ is characteristic for one external branch length in those processes considered. Notice that $\mu^{(\Lambda,n)}$ concerns only the measure $\Lambda\mathbf{1}_{[1/n,1]}$, so it is natural to think about the influences of measures $\Lambda\mathbf{1}_{[1/n,1]}$ and $\Lambda\mathbf{1}_{[0,1/n)}$ on one external branch length. More generally, if $\Lambda = \Lambda_1 + \Lambda_2$, how can we evaluate each influence on the construction of the whole Λ coalescent? To this aim, we introduce in the next section the "measure division construction" of a Λ coalescent. The idea of this construction can be at least tracked back to [2] where the authors consider a coupling of two finite measure on $[0,1]$.

Let Γ be a signed measure on $[0,1]$ (signed measures include measures). According to Hahn decomposition theorem (see [3] for example), every signed measure Γ has a decomposition P and N which are two subsets of $[0,1]$ such that

- $P\bigcup N = [0,1]$, and $P\bigcap N = \phi$.
- $\forall F \subset P$, we have $\Gamma(F) \geq 0$.
- $\forall F \subset N$, we have $\Gamma(F) \leq 0$.

We define by $\mu^+(\Gamma)$(resp. $\mu^-(\Gamma)$) the measure such that for any $F \subset [0,1]$, we have $\mu^+(\Gamma)(F) = \Gamma(P\bigcap F)$ (resp. $\mu^-(\Gamma)(F) = -\Gamma(N\bigcap F)$). Both $\mu^+(\Gamma)$ and $\mu^-(\Gamma)$ are measures. We denote by $|\Gamma| := \mu^+(\Gamma) + \mu^-(\Gamma)$ the variation of Γ. Let Γ_1 and Γ_2 be two signed measures and we denote by $\Gamma_1 \leq \Gamma_2$, if $\Gamma_2 - \Gamma_1$ is a measure. Now we turn to the main results.

Theorem 1.1. *Let Γ be a signed measure and $|\Gamma|$ be the variation of Γ. We assume that $\int_0^1 -\ln(1-x)x^{-2}|\Gamma|(dx) < +\infty$ and $\Lambda^{leb} + \Gamma$ is a measure. Then the Λ coalescent with $\Lambda = \Lambda^{leb} + \Gamma$ satisfies:*

$$\tau^{(\Lambda,n)} - \ln\ln n \overset{(d)}{\to} G, \tag{4}$$

where G is standard Gumbel distributed, i.e. with distribution function $exp(-exp(-x)), x \in \mathbb{R}$.

Remark 1.1. We can apply this theorem to the $Beta(1,b)$ coalescents with $b > 0$ to get:

$$\frac{1}{b}(\tau^{(\Lambda,n)} - \ln\ln n) \overset{(d)}{\to} G.$$

This closes an open problem posted by Gnedin et al [16].

Theorem 1.2. *If Λ satisfies:*

$$\lim_{n\to+\infty} \frac{g_n^{(\Lambda)}}{n\mu^{(\Lambda,n)}} = 0, \tag{5}$$

then $\mu^{(\Lambda,n)}T^{(\Lambda,n)} \overset{(d)}{\to} Exp(1)$.

Remark 1.2. Condition (5) implies that $\Lambda(\{0\}) = 0$. Indeed, if $\Lambda(\{0\}) > 0$, then $g_n^{(\Lambda)} \geq \binom{n}{2}\Lambda(\{0\})$ and $\mu^{(\Lambda,n)} \leq n\Lambda((0,1])$. Then (5) is invalid.

Examples: We give a short list of typical examples satisfying condition (5) which are processes without proper frequencies or look like "around" Bolthausen-Szitman coalescent.

Ex 1: $\int_0^1 x^{-1}\Lambda(dx) < +\infty$: It suffices to prove that $\lim_{n\to+\infty} \frac{g_n^{(\Lambda)}}{n} = 0$. Recalling the expression (3) of $g_n^{(\Lambda)}$, we have, for $n \geq 2$,

$$\begin{aligned}
\frac{g_n^{(\Lambda)}}{n} &= \frac{\int_0^1 (1-(1-x)^n - nx(1-x)^{n-1})x^{-2}\Lambda(dx)}{n} \\
&= \frac{\int_{1/n}^1 (1-(1-x)^n - nx(1-x)^{n-1})x^{-2}\Lambda(dx)}{n} + \frac{\int_0^{1/n} (1-(1-x)^n - nx(1-x)^{n-1})x^{-2}\Lambda(dx)}{n} \\
&\leq \frac{\int_{1/n}^1 x^{-2}\Lambda(dx)}{n} + \frac{\int_0^{1/n} n^2\Lambda(dx)}{n}. \qquad (6)
\end{aligned}$$

The second term $\frac{\int_0^{1/n} n^2\Lambda(dx)}{n} = \int_0^{1/n} n\Lambda(dx) \leq \int_0^{1/n} x^{-1}\Lambda(dx) \to 0$. For the first term, let $\epsilon > 0$ and $M = 1/\epsilon$, then

$$\begin{aligned}
\frac{\int_{1/n}^1 x^{-2}\Lambda(dx)}{n} &= \frac{\int_{M/n}^1 x^{-2}\Lambda(dx)}{n} + \frac{\int_{1/n}^{M/n} x^{-2}\Lambda(dx)}{n} \\
&\leq \frac{\int_{M/n}^1 x^{-1}\Lambda(dx)}{M} + \int_{1/n}^{M/n} x^{-1}\Lambda(dx) \\
&\leq \epsilon \int_0^1 x^{-1}\Lambda(dx) + \int_{1/n}^{M/n} x^{-1}\Lambda(dx).
\end{aligned}$$

Notice that $\epsilon\int_0^1 x^{-1}\Lambda(dx)$ can be arbitrarily small and $\int_{1/n}^{M/n} x^{-1}\Lambda(dx)$ tends to 0 as n tends to $+\infty$. Then we get that $\frac{\int_{1/n}^1 x^{-2}\Lambda(dx)}{n}$ tends to 0. Hence if $\int_0^1 x^{-1}\Lambda(dx) < +\infty$ is satisfied, we get condition (5).

Ex 2: Bolthausen-Sznitman coalescent: In this case, it is straightforward to prove that $g_n^{(\Lambda)} = n-1$ and $\mu^{(\Lambda,n)} = \ln n$, then $\lim_{n\to+\infty} \frac{g_n^{(\Lambda)}}{n\mu^{(\Lambda,n)}} = \lim_{n\to+\infty} \frac{n-1}{n\ln n} = 0$.

Ex 3: Λ has a density function f_Λ on $[0, r)$ where $0 < r < 1$ and there exists a positive number M such that $f_\Lambda < M$ on $[0, r)$: This kind of processes can be considered as being *dominated* by the Bolthausen-Sznitman coalescent.

If $\int_0^1 x^{-1}\Lambda(dx) < +\infty$, we turn back to the first example. If $\int_0^1 x^{-1}\Lambda(dx) = +\infty$, then we have $g_n^{(\Lambda)} \leq 2M(n-1)$ for n large enough, hence $\limsup_{n\to+\infty} \frac{g_n^{(\Lambda)}}{n\mu^{(\Lambda,n)}} \leq \lim_{n\to+\infty} \frac{2M(n-1)}{n\mu^{(\Lambda,n)}} = 0$. It turns out that this kind of coalescents also satisfies condition (5).

Ex 4: Λ has a density function $f_\Lambda(x) = p(\ln\frac{1}{x})^q$ on $[0, r)$ where $0 < r < 1$ and p, q are positive numbers: Using (6), we have

$$\frac{g_n^{(\Lambda)}}{n\mu^{(\Lambda,n)}} \leq \frac{\int_{1/n}^1 x^{-2}\Lambda(dx)}{n\mu^{(\Lambda,n)}} + \frac{\int_0^{1/n} n^2\Lambda(dx)}{n\mu^{(\Lambda,n)}}, \forall n \geq 2.$$

For two real sequences $(x_n)_{n\geq1}, (y_n)_{n\geq1}$, we write $x_n \asymp y_n$, if there exist two positive constants c, C such that $cy_n \leq x_n \leq Cy_n$ for n large enough. Then it is not difficult to find out that $\mu^{(\Lambda,n)} \asymp (\ln n)^{q+1}$, $\int_{1/n}^1 x^{-2}\Lambda(dx) \asymp n(\ln n)^q$, $\int_0^{1/n} n^2\Lambda(dx) \asymp n(\ln n)^q$. Hence we get $\frac{g_n^{(\Lambda)}}{n\mu^{(\Lambda,n)}} \asymp \frac{1}{\ln n} \to 0$.

Theorem 1.3. *If Λ satisfies condition (5) and $\int_0^1 x^{-1}\Lambda(dx) = +\infty$, then we have:*

$$\mu^{(\Lambda,n)}(T_1^{(\Lambda,n)}, T_2^{(\Lambda,n)}, \cdots, T_n^{(\Lambda,n)}, 0, 0, \cdots) \stackrel{(d)}{\to} (e_1, e_2, \cdots), \qquad (7)$$

where $(e_i)_{i\in\mathbb{N}}$ *are independently distributed as* $Exp(1)$.

Remark 1.3. The same result has been proved for Bolthausen-Sznitman coalescent in [9]. The authors have used a moment method. We can apply this theorem to Example 4 and Example 3 with $\int_0^1 x^{-1}\Lambda(dx) = +\infty$.

The following three corollaries have also been proved for Bolthausen-Sznitman coalescent (see [9], [12], [16]).

Corollary 1.4. *If* Λ *satisfies condition (5), then for any* $r \in \mathbb{R}^+$,

$$\lim_{n\to+\infty} \mathbb{E}[(\mu^{(\Lambda,n)}T^{(\Lambda,n)})^r] = \mathbb{E}[e_1^r],$$

where e_1 is distributed as $Exp(1)$. Moreover, if $\int_0^1 x^{-1}\Lambda(dx) = +\infty$, then for any $k \in \mathbb{N}$ and any $(r_1, r_2, \cdots, r_k) \in (\mathbb{R}^+)^k$, we have:

$$\lim_{n\to+\infty} \mathbb{E}[\prod_{i=1}^{k}(\mu^{(\Lambda,n)}T_i^{(\Lambda,n)})^{r_i}] = \mathbb{E}[\prod_{i=1}^{k} e_i^{r_i}],$$

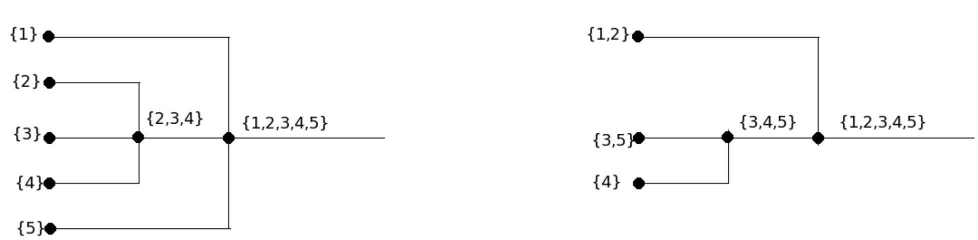
where $(e_i)_{1\leq i\leq k}$ are independently distributed as $Exp(1)$.

Corollary 1.5. *If* Λ *satisfies condition (5) and* $\int_0^1 x^{-1}\Lambda(dx) = +\infty$, *then the total external branch length* $L_{ext}^{(\Lambda,n)}$ *satisfies:* $\mu^{(\Lambda,n)}L_{ext}^{(\Lambda,n)}/n$ *converges in probability to* 1.

Corollary 1.6. *If* Λ *satisfies condition (5) and* $\int_0^1 x^{-1}\Lambda(dx) = +\infty$, *then the total branch length* $L^{(\Lambda,n)}$ *satisfies:* $\mu^{(\Lambda,n)}L^{(\Lambda,n)}/n$ *converges in probability to* 1.

Remark 1.4. In fact, we will prove that $\lim_{n\to+\infty} \mathbb{E}[\mu^{(\Lambda,n)}L^{(\Lambda,n)}/n] = 1$. Notice that Corollary 1.4 gives $\lim_{n\to+\infty} \mathbb{E}[\mu^{(\Lambda,n)}L_{ext}^{(\Lambda,n)}/n] = 1$. Hence we deduce this corollary using Corollary 1.5.

1.2. **Organization.** In section 2, we introduce the main object of this paper: the measure division construction. In this section, we define the restriction by the smallest element which serves as a preliminary step to measure division construction. In section 3, we use the established construction method to study the MRCA of a Boltausen-Sznitman process perturbed by a signed measure. In section 4, we study the lengths of branchs of those processes satisfying the condition (5).



(a) $\Pi^{(\Lambda,5)}$

(b) A restriction by the smallest element of $\Pi^{(\Lambda,5)}$ from $\{\{1\},\cdots,\{5\}\}$ to $\{\{1,2\},\{3,5\},\{4\}\}$

FIGURE 1. Restriction by the smallest element

2. Measure division construction

2.1. **Restriction by the smallest element.** Let $\xi_n = \{A_1, \cdots, A_{|\xi_n|}\}$, $\chi_n = \{B_1, \cdots, B_{|\chi_n|}\}$ be two partitions of $\{1, 2, \cdots, n\}$. We define s_i^A (resp. s_i^B) as the smallest number in the block A_i (resp. B_i). We define also the notation $\xi_n \preceq \chi_n$, if $|\chi_n| \leq |\xi_n|$ and for any $1 \leq i \leq |\chi_n|, B_i = \bigcup_{j \in I_i} A_j$, where $\{I_i\}_{1 \leq i \leq |\chi_n|}$ is a partition of $\{1, 2, \cdots, |\xi_n|\}$. Roughly speaking, ξ_n is finer than χ_n.

If $\xi_n \preceq \chi_n$, we define the stochastic process $\bar{\Pi}^{(\Lambda,\chi_n)}$, called the restriction by the smallest element of $\Pi^{(\Lambda,\xi_n)}$ from ξ_n to χ_n:

- $\bar{\Pi}^{(\Lambda,\chi_n)}(0) = \chi_n$;
- For any $t \geq 0$, if $\Pi^{(\Lambda,\xi_n)}(t) = \{D_i\}_{1 \leq i \leq |\Pi^{(\Lambda,\xi_n)}|(t)}$, where D_i denotes a block, then
$$\bar{\Pi}^{(\Lambda,\chi_n)}(t) = \{ \bigcup_{s_j^B \in D_i} B_j \}_{1 \leq i \leq |\Pi^{(\Lambda,\xi_n)}|(t)},$$
where the empty sets in $\bar{\Pi}^{(\Lambda,\chi_n)}(t)$ are removed.

Notice that the restriction by the smallest element is also defined from path to path (see Figure 1).

Lemma 2.1. $\bar{\Pi}^{(\Lambda,\chi_n)}$ *has the same distribution as* $\Pi^{(\Lambda,\chi_n)}$.

Proof. Every block in χ_n is identified by its smallest element which belongs to a unique block in ξ_n. Hence for any B_i in χ_n, there exists a unique A_{τ_i} such that $A_{\tau_i} \in \xi_n$, $A_{\tau_i} \subset B_i$ and $s_{\tau_i}^A = s_i^B$ with $\tau_i \in \{1, 2, \cdots, |\xi_n|\}$. Let $\chi_n' = \{A_{\tau_i}\}_{1 \leq i \leq |\chi_n|}$ and define a new process $\hat{\Pi}^{(\Lambda,\chi_n')}$ as follows:

- $\hat{\Pi}^{(\Lambda,\chi_n')}(0) = \chi_n'$.
- For any $t \geq 0$, if $\Pi^{(\Lambda,\xi_n)}(t) = \{D_i\}_{1 \leq i \leq |\Pi^{(\Lambda,\xi_n)}|(t)}$, then
$$\hat{\Pi}^{(\Lambda,\chi_n')}(t) = \{ \bigcup_{s_{\tau_j}^A \in D_i} A_{\tau_j} \}_{1 \leq i \leq |\Pi^{(\Lambda,\xi_n)}|(t)},$$
where the empty sets in $\hat{\Pi}^{(\Lambda,\chi_n')}(t)$ are removed.

It is easy to see that $\hat{\Pi}^{(\Lambda,\chi_n')}$ is a restriction of $\Pi^{(\Lambda,\xi_n)}$ from ξ_n to χ_n'. By the consistency property, we get $\hat{\Pi}^{(\Lambda,\chi_n')} \stackrel{(d)}{=} \Pi^{(\Lambda,\chi_n')}$. Because of the construction of $\hat{\Pi}^{(\Lambda,\chi_n')}$ and $\bar{\Pi}^{(\Lambda,\chi_n)}$, what is determinant is the smallest element in each block. Hence to obtain $\bar{\Pi}^{(\Lambda,\chi_n)}$ from $\hat{\Pi}^{(\Lambda,\chi_n')}$, at time 0, one needs to complete every A_{τ_i} by some other numbers larger than $s_{\tau_i}^A$ to get B_i and then follow the evolution of $\hat{\Pi}^{(\Lambda,\chi_n')}$. It turns out that $\bar{\Pi}^{(\Lambda,\chi_n)}$ is a coalescent process with initial value χ_n. Hence we can conclude. □

2.2. **Measure division construction.** Let $\Lambda, \Lambda_1, \Lambda_2$ be three measures such that $\Lambda = \Lambda_1 + \Lambda_2$ and $\int_0^1 \Lambda(dx) < +\infty$.

We denote by $\Pi_{1,2}^{(\Lambda,n)} := (\Pi_{1,2}^{(\Lambda,n)}(t))_{t \geq 0}$ the stochastic process constructed by the measure division construction using Λ_1 and Λ_2. Here the index $(1,2)$ is for $\Lambda = \Lambda_1 + \Lambda_2$ with Λ_1 called "noise measure" and Λ_2 "main measure".

- Step 0: Given a realization or a path of $\Pi^{(\Lambda_1,n)}$, we set $\Pi_{1,2}^{(\Lambda,n)}(t) = \Pi^{(\Lambda_1,n)}(t)$, for any $t \geq 0$. We set also $t_0 = 0$.
- Step 1: Let $t_1, t_2, \cdots$ be the coalescent times after t_0 of the given path of $\Pi_{1,2}^{(\Lambda,n)}$ (if there is no collision after t_0, we set $t_i = +\infty, i \geq 1$). Within $[t_0, t_1)$, $\Pi_{1,2}^{(\Lambda,n)}$ is constant. Then we run an independent Λ_2 coalescent with initial value $\Pi_{1,2}^{(\Lambda,n)}(t_0)$ from time t_0.
  - If the Λ_2 coalescent has no collision on $[t_0, t_1)$, we pass to $[t_1, t_2)$. Similarly, we construct another independent Λ_2 coalescent with initial value $\Pi_{1,2}^{(\Lambda,n)}(t_1)$ from time t_1, and so on.
  - Otherwise, we go to the next step.
- Step 2: If finally within $[t_{i-1}, t_i)$, the related independent Λ_2 coalescent has its first collision at time t_* and its value at t_* is ξ. We set the new $(\Pi_{1,2}^{(\Lambda,n)}(t))_{t \geq t_*}$ as the restriction by the

smallest element of previous $(\Pi_{1,2}^{(\Lambda,n)}(t))_{t\geq t_*}$ from previous $\Pi_{1,2}^{(\Lambda,n)}(t_*)$ to ξ. Then we go to step 1 taking t_* as the starting time t_0. Notice that by this restriction by the smallest element at time t_*, conditioning on $\{\Pi_{1,2}^{(\Lambda,n)}(t_*) = \xi\}$, $(\Pi_{1,2}^{(\Lambda,n)}(t))_{t\geq t_*}$ has the same distribution as a Λ_1 coalescent from time t_* with initial value ξ, due to Lemma 2.1.

Remark 2.1.
- The measure division construction works path by path and $\Pi_{1,2}^{(\Lambda,n)}$ is constructed by following the time. We also find out that this construction is based on $\Pi^{(\Lambda_1,n)}$ which is random. Hence this construction is established in a random environment.
- If we take $\Lambda_1 = 0$ as noise measure and $\Lambda_2 = \Lambda$ as main measure, then $\Pi^{(\Lambda_1,n)}(t) = \{\{1\},\{2\},\cdots,\{n\}\}$ for any $t \geq 0$ and $\Pi_{1,2}^{(\Lambda,n)} \overset{(d)}{=} \Pi^{(\Lambda,n)}$.

Theorem 2.2. *Let Λ, Λ_1 and Λ_2 be three measures and $\Lambda = \Lambda_1 + \Lambda_2$. Then we have $\Pi_{1,2}^{(\Lambda,n)} \overset{(d)}{=} \Pi^{(\Lambda,n)}$.*

Proof. Let t be a coalescent time of $\Pi_{1,2}^{(\Lambda,n)}$. We consider the time of the next coalescence and the value at that moment. In the measure division construction of $\Pi_{1,2}^{(\Lambda,n)}$, we can see appearing two independent processes with one a Λ_1 coalescent with initial value $\Pi_{1,2}^{(\Lambda,n)}(t)$ and the other one a Λ_2 coalescent with initial value $\Pi_{1,2}^{(\Lambda,n)}(t)$ from time t. The process $\Pi_{1,2}^{(\Lambda,n)}$ gets the next coalescence whenever one of them firstly encounters a coalescence and picks up the value of the process at that moment. Then we follow the same procedure from the new coalescent time of $\Pi_{1,2}^{(\Lambda,n)}$. It is easy to see that $\Pi_{1,2}^{(\Lambda,n)}$ behaves in the same way as $\Pi^{(\Lambda,n)}$. Hence we can conclude. □

Remark 2.2. The theorem shows that if we exchange the noise measure and the main measure, the distribution of the process is not changed and is uniquely determined by their sum.

Remark 2.3. The measure division construction also works for more than two measures. If there are $k(k \geq 2)$ measures $\{\Lambda_i\}_{1\leq i\leq k}$ and $\Lambda = \sum_{i=1}^{k} \Lambda_i$, one can get a stochastic process by first giving a realization of $\Pi^{(\Lambda_1,n)}$ which will be modified by Λ_2 in the way described in the measure division construction, and then we apply Λ_3 on the modified process, etc. The equivalence in distribution can be obtained in a recursive way.

We give two corollaries to show some properties of the measure division construction.

Corollary 2.3. *Let Π_1, Π_2 be two given coalescent paths of respectively two whatever coalescent processes , and $|\Pi_1| \leq |\Pi_2| < +\infty$ at any time. We apply a finite measure Λ to them using measure division construction. Then the time back to MCRA related to Π_1 is stochastically dominated by that of Π_2.*

Proof. At first, we rename the blocks of $\Pi_1(0)$ and of $\Pi_2(0)$ as that $\Pi_1(0) = \{\{1\},\{2\},\cdots,\{|\Pi_1|(0)\}\}$ and $\Pi_2(0) = \{\{1\},\{2\},\cdots,\{|\Pi_2|(0)\}\}$. Then Π_1 (resp. Π_2) can be regarded as a coalescent path descending from $\{\{1\},\{2\},\cdots,\{|\Pi_1|(0)\}\}$ (resp. $\{\{1\},\{2\},\cdots,\{|\Pi_2|(0)\}\}$). We will modify Π_1(resp. Π_2) to Π_1' (resp. Π_2') using the finite measure Λ and the measure division construction. At first we set $\Pi_1' = \Pi_1$ (resp. $\Pi_2' = \Pi_2$). Let $s_1 = \min\{t|\Pi_1'(t-) \neq \Pi_1'(t), \quad \text{or} \quad \Pi_2'(t-) \neq \Pi_2'(t)\}$. Then we run an independent Λ coalescent with initial value $\Pi_2(0)$ (renamed).

- Step 1: If the Λ coalescent has its first coalescence within $[0, s_1)$, we apply it to Π_2' in the same way as described in the measure division construction and also to Π_1' using restriction. Hence we get the new Π_1' and Π_2' and we do the same thing taking s_1 as a starting point.
- Step 2: Otherwise, we pass to $[s_1, s_2)$, where $s_2 = \min\{t > s_1|\Pi_1'(t-) \neq \Pi_1'(t), \quad \text{or} \quad \Pi_2'(t-) \neq \Pi_2'(t)\}$. We construct another independent Λ coalescent with initial value $\Pi_2'(s_1)$ and go to step 1.

Notice that the modifications are coupled and if we consider single Π_1 or Π_2, due to consistency property, each path is modified following the same procedure as described in the measure division construction (especially for Π_1). It is evident to see that the coupling let Π_1' arrive to its MRCA earlier than Π_2' to its own MRCA. Then we can conclude. □

We denote by "$\ll$" the stochastic domination between two real random variables. The following corollary is the same as Lemma 3.2 in [2]. But we prove it again in our way.

Corollary 2.4. *Let Λ_1, Λ_2 be two finite measures such that $\Lambda_1 \leq \Lambda_2$, then we have $\tau^{(\Lambda_2,n)} \ll \tau^{(\Lambda_1,n)}$.*

Proof. The variable $\tau^{(\Lambda_1,n)}$ can be viewed as the time back to the MRCA of the measure division constructed process with the noise measure Λ_1 and the main measure null. The variable $\tau^{(\Lambda_2,n)}$ can be viewed as that of the process with noise measure Λ_1 and main measure $\Lambda_2 - \Lambda_1$. Since the measure division construction can be established from path to path for both using the same realization of $\Pi^{(\Lambda_1,n)}$, so we can conclude. The proof can also be done using Corollary 2.3. $\square$

2.3. **A tripling.** We often have some results on the coalescent related to a special measure, for example, the *Beta* coalescent. When the process is perturbed by a noise measure, we would wonder whether this damage is negligible. One example is to estimate the number of blocks of the coalescent related to the noise measure after a certain time (if this noise measure is a signed measure, one can study its variation). To this aim, we use the tool of tripling.

Tripling: Notice that $\Pi^{(\Lambda,n)}$ encounters its first collision after time $e_1^{(n)}$, which is a random variable. At this collision, the number of blocks is reduced to $n - W_1^{(n)}$, where $W_1^{(n)}$ is random. Then we add $W_1^{(n)}$ new blocks (these blocks can contain any numbers belonging to $\{n+1, n+2, \cdots\}$) and consider the whole new n ones. By the consistency property, the evolution of the original $n - W_1^{(n)}$ blocks can be embedded into that of the new n blocks, i.e. after time $e_2^{(n)}$, we have the collision in the new n blocks whose total number is reduced to $n - W_2^{(n)}$ and we can calculate the distribution of the number of blocks coalesced among the original $n - W_1^{(n)}$ blocks (we call any block containing at least one of $\{1, 2, \cdots, n\}$ as "original block" and it is very possible that nothing happens for the $n - W_1^{(n)}$ blocks). Then we add again new blocks containing different elements to have another n ones. This procedure is stopped when every element of $\{1, 2, \cdots, n\}$ is contained in one block. By the definition of Λ coalescent, $(e_i^{(n)})_{i\geq 1}$ are independent exponential random variables with parameter $g_n^{(\Lambda)}$ and $(W_i^{(n)})_{i\geq 1}$ are i.i.d copies of $X_1^{(\Lambda,n)}$.

The above procedure gives a tripling of $(e_i^{(n)})_{i\geq 1}$, $(W_i^{(n)})_{i\geq 1}$ and $\Pi^{(\Lambda,n)}$. We define $V_i^{(n)} := \sum_{j=1}^{i} e_j^{(n)}, i \in \mathbb{N}$. Then we have the following proposition:

Proposition 2.5. *Suppose that $(e_i^{(n)})_{i\geq 1}$, $(W_i^{(n)})_{i\geq 1}$ and $\Pi^{(\Lambda,n)}$ are tripled, then at any time $t \geq 0$, we have*

$$n - \sum_{i=0}^{N(\Lambda,n,t)} W_i^{(n)} \leq |\Pi^{(\Lambda,n)}|(t), \tag{8}$$

where $N(\Lambda, n, t) := card\{i | V_i^{(n)} \leq t\}$, which is Poisson distributed with parameter $g_n^{(\Lambda)} t$ and independent of $(W_i^{(n)})_{i\geq 1}$. Meanwhile,

$$\mathbb{E}[W_i^{(n)}] = \frac{n \int_0^1 (1-(1-x)^{n-1}) x^{-1} \Lambda(dx)}{g_n^{(\Lambda)}} - 1, \textit{and} \quad \mathbb{E}[(W_i^{(n)})^2] = \frac{n(n-1)\int_0^1 \Lambda(dx)}{g_n^{(\Lambda)}} - \mathbb{E}[W_i^{(n)}]. \tag{9}$$

Proof. The number of is within $[0, t]$ follows the Poisson distribution with parameter $g_n^{(\Lambda)} t$. Due to the tripling, at any time $V_i^{(n)}$ with $0 \leq V_i^{(n)} \leq t$, the decrease of number of blocks (i.e. $|\Pi^{(\Lambda,n)}|(V_i^{(n)}-) - |\Pi^{(\Lambda,n)}|(V_i^{(n)})$) among original blocks is less than or equal to $W_i^{(n)}$. Hence we get (8). Notice that $W_i^{(n)} \overset{(d)}{=} X_1^{(\Lambda,n)}$, then (9) is a consequence of two equalities in [8] with Eq (17) for the first one and p.1007 for the second one. $\square$

3. Application 1 : *Theorem*1.1.

3.1. **Annihilator process.** In Theorem 1.1, the main measure Λ^{leb} is perturbed by a signed measure whose variation is related to a coalescent process without proper frequencies. To estimate the damage made by the noise measure, we first recall the definition of annihilator processes.

Let Λ be a finite measure on $(0,1]$ and $\int_0^1 x^{-1}\Lambda(dx) < +\infty$. The annihilator process related to Λ, denoted by $\tilde{\Pi}^{(\Lambda,n)}$, was introduced in [17]. It is a càdlàg continuous time Markov process taking sets of blocks which are subsets of $\{1,2,\cdots,n\}$ as values. At time 0, $\tilde{\Pi}^{(\Lambda,n)}(0) = \{\{1\},\{2\},\cdots,\{n\}\}$. As time goes by, we remove several blocks uniformly until its value equals empty. More precisely, let $|\tilde{\Pi}^{(\Lambda,n)}|$ be the block counting process of $\tilde{\Pi}^{(\Lambda,n)}$ and if at time t, $|\tilde{\Pi}^{(\Lambda,n)}|(t) = b$ with $b \geq 1$, then it encounters the next remove after an exponential time with parameter:

$$\tilde{g}_b^{(\Lambda)} := \sum_{k=1}^{b}\int_0^1 \binom{b}{k} x^{k-2}(1-x)^{b-k}\Lambda(dx),$$

and the probability to remove a group of $k(1 \leq k \leq b)$ blocks with the others unchanged is:

$$\frac{\int_0^1 x^{k-2}(1-x)^{b-k}\Lambda(dx)}{\tilde{g}_b^{(\Lambda)}}.$$

There are two constructions given in [17]. Both of them concern the coupling between the coalescent and annihilator process.

First construction: The general case with $\int_0^1 x^{-1}\Lambda(dx) < +\infty$. This construction is mainly due to the remark in section 3 of [17]. Let $(\tilde{S}_t)_{t\geq 0}$ be a subordinator with the Laplace transform

$$\mathbb{E}[e^{-\lambda\tilde{S}_t}] = e^{-t\Phi(\lambda)}, t \geq 0, \lambda \geq 0, \tag{10}$$

where the Laplace exponent $\Phi(\lambda) = \int_0^1 (1-(1-x)^\lambda)x^{-2}\Lambda(dx)$. Define $S_t := e^{-\tilde{S}_t}$. Then S_t is a non-increasing non-negative process with value 1 at time 0. Let $(u_i)_{1\leq i\leq n}$ be n independent uniform random variables on $[0,1]$, then $(\tilde{\Pi}^{(\Lambda,n)}(t))_{t\geq 0} \overset{(d)}{=} \{i|u_i \leq S_t, 1 \leq i \leq n\}$. Due to a coupling not recalled here between $\tilde{\Pi}^{(\Lambda,n)}$ and $\Pi^{(\Lambda,n)}$, we can find a probability space where we have for any $t \geq 0$,

$$|\tilde{\Pi}^{(\Lambda,n)}|(t) \leq |\Pi^{(\Lambda,n)}|(t). \tag{11}$$

Second construction: The case with $\int_0^1 x^{-2}\Lambda(dx) < +\infty$. This one is due to section 2 of [17]. Here we recall the precise coupling. Let $(\eta_i^{(\Lambda)})_{i\geq 1}$ be independent random variables following the distribution of $\frac{x^{-2}\Lambda(dx)}{\int_0^1 x^{-2}\Lambda(dx)}$ and $(e_i^{(\Lambda)})_{i\geq 1}$ i.i.d copies of $Exp(\int_0^1 x^{-2}\Lambda(dx))$. At time 0, we have n singletons from 1 to n which are called *primary*. At random time $e_1^{(\Lambda)}$, conditioning on $\eta_1^{(\Lambda)}$, every block is independently marked "Head" with probability $\eta_1^{(\Lambda)}$ and "Tail" with probability $1-\eta_1^{(\Lambda)}$. Those blocks marked "Head" are merged into a bigger block called *secondary*, provided that there are at least two "Head"s. If there is only one "Head", we call this block *secondary* and do nothing else. Then after another time $e_2^{(\Lambda)}$, we do the same thing, and so on. This construction gives a coupling between the whole block process and *secondary* block process, with the former being $\Pi^{(\Lambda,n)}$ and the latter $\tilde{\Pi}^{(\Lambda,n)}$. It is evident to see that, in this construction, $|\tilde{\Pi}^{(\Lambda,n)}|(t) \leq |\Pi^{(\Lambda,n)}|(t)$, for any $t \geq 0$.

For $i \geq 1$, we call $\sum_{j=1}^i e_j^{(\Lambda)}$ the *i-th marking time* and $(L_j^{(\Lambda,n,i)}, j \geq 1)$ the *i-th selection time* of $\{j\}$ when $\{j\}$ is marked "Head" for the i-th time (we set $L_j^{(\Lambda,n,i)} = +\infty$, if $\{j\}$ has no *i-th selection time*). Notice that $(L_j^{(\Lambda,n,1)})_{1\leq j\leq n}$ are exchangeable. It is easy to see that $L_1^{(\Lambda,n,1)}$ is exponentially distributed with parameter $\int_0^1 x^{-1}\Lambda(dx)$. It is shown by the construction that

$$L_1^{(\Lambda,n,1)} \leq T_1^{(\Lambda,n)}. \tag{12}$$

Remark 3.1. If $\int_0^1 x^{-2}\Lambda(dx) < \infty$, we define the probability $P^{(\Lambda,n)}$ for singleton $\{1\}$ of $\Pi^{(\Lambda,n)}$ not to be coalesced at its *1-st selection time* and define also $(\Delta_i^{(\Lambda)})_{i\geq 1} := (\eta_i^{(\Lambda)}\Pi_{j=1}^{i-1}(1-\eta_j^{(\Lambda)}))_{i\geq 1}$. Notice

that the *1-st selection time* of singleton $\{1\}$ is an exponential time with parameter $\int_0^1 x^{-1}\Lambda(dx)$. To see whether singleton $\{1\}$ is coalesced or not at its *1-st selection time* depends on whether there exists at least another block marked "Head" at the same time or not. The construction shows that at the *1-st selection time* of singleton $\{1\}$, there are more blocks waiting to be marked in coalescent case than in annihilator case. Hence $P^{(\Lambda,n)}$ is less than or equal to the probability for singleton $\{1\}$ of $\tilde{\Pi}^{(\Lambda,n)}$ not to be coalesced at its *1-st selection time* which equals exactly $\sum_{i=1}^{+\infty}\mathbb{E}[\Delta_i^{(\Lambda)}(1-\Delta_i^{(\Lambda)})^{n-1}]$. In short, $P^{(\Lambda,n)} \leq \sum_{i=1}^{+\infty}\mathbb{E}[\Delta_i^{(\Lambda)}(1-\Delta_i^{(\Lambda)})^{n-1}]$.

Lemma 3.1. *Let $\Pi^{(\Lambda,n)}$ be a coalescent with $\int_0^1 -\ln(1-x)x^{-2}\Lambda(dx) < +\infty$. Then for any $r>0$, there exists $m>0$, such that*

$$\lim_{n\to+\infty}\mathbb{P}(|\Pi^{(\Lambda,n)}|((\ln\ln n)^r) \geq ne^{-m(\ln\ln n)^r}) = 1.$$

Proof. Notice that Λ satisfies $\int_0^1 x^{-1}\Lambda(dx) < +\infty$. Hence we can use the first construction above where we pick up the notations. The dominance (11) gives that it suffices to prove

$$\lim_{n\to+\infty}\mathbb{P}(|\tilde{\Pi}^{(\Lambda,n)}|((\ln\ln n)^r) \geq ne^{-m(\ln\ln n)^r}) = 1.$$

A derivation of (10) on λ gives that

$$\mathbb{E}[\tilde{S}_t e^{-\lambda\tilde{S}_t}] = e^{-t\Phi(\lambda)}\int_0^1 -t(1-x)^\lambda\ln(1-x)x^{-2}\Lambda(dx).$$

Then let λ tend to 0 to get $\mathbb{E}[\tilde{S}_t] = \int_0^1 -t\ln(1-x)x^{-2}\Lambda(dx) < +\infty$. Hence, it follows that $\frac{\tilde{S}_t}{t}$ converges in probability to $\mathbb{E}[\tilde{S}_1]$ due to the law of large number. In particular, we have $\frac{\tilde{S}_{(\ln\ln n)^r}}{(\ln\ln n)^r}$ converging in probability to $\mathbb{E}[\tilde{S}_1]$ which implies that $\frac{S_{(\ln\ln n)^r}}{e^{-\mathbb{E}[\tilde{S}_1](\ln\ln n)^r}}$ converges in probability to 1.

Let $X_i := \mathbf{1}_{u_i \leq S_{(\ln\ln n)^r}}, 1 \leq i \leq n$. Hence given $p := S_{(\ln\ln n)^r}$, $(X_i)_{1\leq i\leq n}$ are n independent Bernoulli variables with parameter p. We get, due to the subordinator based construction,

$$\sum_{i=1}^n X_i \overset{(d)}{=} |\tilde{\Pi}^{(\Lambda,n)}|((\ln\ln n)^r). \tag{13}$$

Chernoff's inequality says that for any $0 \leq \epsilon \leq p$, with p given,

$$\mathbb{P}(\frac{\sum_{i=1}^n X_i}{n} \leq p-\epsilon|p) \leq e^{-D(p-\epsilon||p)n},$$

where $D(x||y) = x\ln\frac{x}{y} + (1-x)\ln\frac{1-x}{1-y}, 0 \leq x \leq y \leq 1$ (at the boundary, we set $D(0||0) = D(1||1) = 0$ and let $D(x||y)$ extend continuously to other points). We choose $\epsilon = p - p^2$, then $D(p-\epsilon||p) = D(p^2||p) = p^2\ln p + (1-p^2)\ln\frac{1-p^2}{1-p} \geq p^2\ln p + (1-p^2)(-p^2+p-\frac{p^2}{2}) = p(1-p^2+p\ln p) - \frac{3p^2}{2}(1-p^2)$.

Hence conditioning on p, we get

$$\mathbb{P}(\frac{\sum_{i=1}^n X_i}{n} \leq p^2|p) \leq e^{-(p(1-p^2+p\ln p|p)-\frac{3p^2}{2}(1-p^2))n}, \tag{14}$$

where the right-hand term converges in probability to 0, since $\frac{p}{e^{-\mathbb{E}[\tilde{S}_1](\ln\ln n)^r}} = \frac{S_{(\ln\ln n)^r}}{e^{-\mathbb{E}[\tilde{S}_1](\ln\ln n)^r}}$ converges in probability to 1. Then for $m \geq 2\mathbb{E}[\tilde{S}_1]$,

$$\lim_{n\to+\infty}\mathbb{P}(\frac{\sum_{i=1}^n X_i}{n} \geq e^{-m(\ln\ln n)^r}) = \lim_{n\to+\infty} 1 - \mathbb{E}[\mathbb{P}(\frac{\sum_{i=1}^n X_i}{n} \leq p^2|p)] = 1,$$

We conclude with (13). □

3.2. **Proof of Theorem 1.1.** Let $\Gamma_1 = \mu^-(\Gamma)$, $\Gamma_2 = \mu^+(\Gamma)$, then $\Lambda^{leb} - \Gamma_1 \leq \Lambda = \Lambda^{leb} + \Gamma \leq \Lambda^{leb} + \Gamma_2$. Define $\Pi_1^{(\Lambda,n)}$ (resp. $\Pi_2^{(\Lambda,n)}$) as the coalescent process related to $\Lambda^{leb} - \Gamma_1$ (resp. $\Lambda^{leb} + \Gamma_2$) with the time back to the MRCA $\tau_1^{(\Lambda,n)}$ (resp. $\tau_1^{(\Lambda,n)}$). Thanks to Corollary 2.4,

$$\tau_2^{(\Lambda,n)} \ll \tau^{(\Lambda,n)} \ll \tau_1^{(\Lambda,n)}. \tag{15}$$

Then it suffices to prove that

$$\tau_1^{(\Lambda,n)} - \ln\ln n \stackrel{(d)}{\to} G, \text{and} \quad \tau_2^{(\Lambda,n)} - \ln\ln n \stackrel{(d)}{\to} G. \tag{16}$$

This result was already proved in the case of Boltausen-Sznitman coalescent (or Λ^{leb} coalescent) for which we denote by $\tau^{(*,n)}$ the time back to the MRCA. Comparing the measures, we also have $\tau_2^{(\Lambda,n)} \ll \tau^{(*,n)} \ll \tau_1^{(\Lambda,n)}$.

Let $\tau^{(*,n)}$ be obtained from the measure division constructed $\Pi_{1,2}^{(\Lambda^{leb},n)}$ with noise measure Γ_1 and main measure $\Lambda^{leb} - \Gamma_1$. We define the event $E := \{|\Pi^{(\Gamma_1,n)}|((\ln\ln n)^2) \geq ne^{-m_1(\ln\ln n)^2}\}$, $m_1 \geq 0$. Then thanks to Lemma 3.1, with some m_1, we have

$$\lim_{n\to+\infty} \mathbb{P}(E) = 1. \tag{17}$$

This result implies that the probability for $\min\{|\Pi^{(\Gamma_1,n)}|(t)_{0\leq t\leq(\ln\ln n)^2}\}$ to be larger than or equal to $ne^{-m_1(\ln\ln n)^2}$ tends to 1. The measure division construction of Λ^{leb} coalescent is to apply measure $\Lambda^{leb} - \Gamma_1$ to a given path of $\Pi^{(\Gamma_1,n)}$. While for $\Pi_1^{(\Lambda,n)}$, one needs to apply $\Lambda^{leb} - \Gamma_1$ to a constant path of n singletons (see the second point in Remark 2.1). Thanks to Corollary 2.3, we fix $x > 0$, and n large enough such that $\ln\ln n + x \leq (\ln\ln n)^2$, then we get

$$\begin{aligned}\mathbb{P}(\tau^{(*,n)} \leq \ln\ln n + x) &\leq \mathbb{P}(\tau_1^{(\Lambda,n)} \leq \ln\ln n + x)\\ &\leq \mathbb{P}(\tau^{(*,|\Pi^{(\Gamma_1,n)}|((\ln\ln n)^2))} \leq \ln\ln n + x|E) \leq \mathbb{P}(\tau^{(*,\lfloor ne^{-m_1(\ln\ln n)^2}\rfloor)} \leq \ln\ln n + x),\end{aligned}$$

where the notation $\lfloor s\rfloor := \max\{m|m \leq s, m \in \mathbb{Z}\}, s \in \mathbb{R}$. We define also $\lceil s\rceil := \min\{m|m \geq s, m \in \mathbb{Z}\}, s \in \mathbb{R}$. Notice that

$$\lim_{n\to+\infty} \mathbb{P}(\tau^{(*,n)} \leq \ln\ln n + x) = \mathbb{P}(G \leq x).$$

Hence due to (17) and $\ln\ln(\lfloor ne^{-m_1(\ln\ln n)^2}\rfloor) - \ln\ln n \to 0$, we also have

$$\lim_{n\to+\infty} \mathbb{P}\left(\tau^{(*,\lfloor ne^{-m_1(\ln\ln n)^2}\rfloor)} \leq \ln\ln n + x|E\right) = \mathbb{P}(G \leq x).$$

Then we can conclude for the first convergence in (16). The second convergence in (16) can be deduced in the same way. Hence together with (15), we prove Theorem 1.1.

4. Application 2: under condition (5).

Some notations for this section: Let Λ be a finite measure on $[0,1]$ and $\Lambda_1 = \Lambda \mathbf{1}_{[0,1/n)}$, $\Lambda_2 = \Lambda \mathbf{1}_{[1/n,1]}$; $\mu^{(\Lambda,1/y)} = \int_y^1 x^{-1}\Lambda(dx)$, $g_{1/y}^{(\Lambda)} = \int_0^1 (1-(1-x)^{1/y} - \frac{1}{y}x(1-x)^{1/y-1})x^{-2}\Lambda(dx)$ with $0 < y \leq 1$. Notice that the definitions of $\mu^{(\Lambda,1/y)}$ and $g_{1/y}^{(\Lambda)}$ are consistent with that of $\mu^{(\Lambda,n)}$ and $g_n^{(\Lambda)}$ when $\Lambda(\{0\}) = 0$. These notations help to examine carefully different measures. One thing to notice is that Λ_1 and Λ_2 both depend on n.

Here we are going to prove Theorem 1.2, Theorem 1.3, Corollary 1.5 and Corollary 1.6. Under condition (5), we decompose Λ into Λ_2 and Λ_1. The idea is to construct $\Pi^{(\Lambda,n)}$ using measure division construction with noise measure Λ_1 and main measure Λ_2. At first, we need to show more details implied by condition (5).

Proposition 4.1. *The following two assertions are equivalent:*
(*): Λ satisfies condition (5);
(**): $\Lambda(\{0\}) = 0$ and there exists a càglàd (limit from right, continuous from left) function $f : [0,1] \to [0,1]$, continuous on 0 with $f(0) = 0$ and a constant $C_1 > 0$, such that

$$\mu^{(\Lambda,1/y)} = C_1 exp(\int_y^1 \frac{f(x)}{x} dx)(1 - f(y)), 0 < y \leq 1. \tag{18}$$

Proof. • We first assume that (*) is true. If Λ satisfies (5), then $\Lambda(\{0\}) = 0$ due to Remark 1.2. For $\mu^{(\Lambda,n)} \neq 0$, we have

$$\frac{g_n^{(\Lambda)}}{n\mu^{(\Lambda,n)}} = \frac{\int_0^1 (1-(1-x)^n - nx(1-x)^{n-1})x^{-2}\Lambda(dx)}{n\mu^{(\Lambda,n)}} = I_1^{(n)} + I_2^{(n)},$$

where $I_1^{(n)} = \frac{\int_{1/n}^1 (1-(1-x)^n - nx(1-x)^{n-1})x^{-2}\Lambda(dx)}{n\mu^{(\Lambda,n)}}$, $I_2^{(n)} = \frac{\int_0^{1/n} (1-(1-x)^n - nx(1-x)^{n-1})x^{-2}\Lambda(dx)}{n\mu^{(\Lambda,n)}}$. Notice that for n large, using monotone property, we have $\frac{e-2}{2e}\frac{\int_{1/n}^1 x^{-2}\Lambda(dx)}{n\mu^{(\Lambda,n)}} \leq I_1^{(n)} \leq \frac{\int_{1/n}^1 x^{-2}\Lambda(dx)}{n\mu^{(\Lambda,n)}}$ and $\frac{1}{3}\frac{n\int_0^{1/n}\Lambda(dx)}{\mu^{(\Lambda,n)}} \leq I_2^{(n)} \leq \frac{n\int_0^{1/n}\Lambda(dx)}{\mu^{(\Lambda,n)}}$. Hence condition (5) is equivalent to

$$\lim_{n\to+\infty} \frac{\int_{1/n}^1 x^{-2}\Lambda(dx)}{n\mu^{(\Lambda,n)}} = 0, \text{and} \lim_{n\to+\infty} \frac{n\int_0^{1/n}\Lambda(dx)}{\mu^{(\Lambda,n)}} = 0, \Lambda(\{0\}) = 0. \tag{19}$$

Then we deduce that

$$\lim_{y\to 0+} \frac{\int_0^y \Lambda(dx)}{y\mu^{(\Lambda,1/y)}} = 0, \Lambda(\{0\}) = 0. \tag{20}$$

Indeed, for $1/y > 2$ and $\mu^{(\lfloor 1/y\rfloor)} \neq 0$, we have

$$\frac{\int_0^y \Lambda(dx)}{y\mu^{(\Lambda,1/y)}} = \frac{\int_0^y \Lambda(dx)}{y\int_y^1 x^{-1}\Lambda(dx)} \leq \frac{\int_0^{1/\lfloor 1/y\rfloor}\Lambda(dx)}{\frac{1}{\lceil 1/y\rceil}\int_{1/\lfloor 1/y\rfloor}^1 x^{-1}\Lambda(x)} = \frac{\lceil 1/y\rceil}{\lfloor 1/y\rfloor}\frac{\lfloor 1/y\rfloor\int_0^{1/\lfloor 1/y\rfloor}\Lambda(dx)}{\int_{1/\lfloor 1/y\rfloor}^1 x^{-1}\Lambda(x)} \overset{y\to 0+}{\to} 0.$$

One thing to notice is that $\lim_{y\to 0+} y\mu^{(\Lambda,1/y)} = 0$ is true for any finite Λ. In fact, for any positive number M and $yM < 1$, we have

$$y\mu^{(\Lambda,1/y)} = y\int_y^1 x^{-1}\Lambda(dx) = y\int_{yM}^1 x^{-1}\Lambda(dx) + y\int_y^{yM} x^{-1}\Lambda(dx) \leq \frac{\int_0^1\Lambda(dx)}{M} + \int_y^{yM}\Lambda(dx),$$

where both terms are as small as we want by taking M large enough and y close enough to 0. Looking into details of $\frac{\int_0^y \Lambda(dx)}{y\mu^{(\Lambda,1/y)}}$ when $\mu^{(\Lambda,1/y)} \neq 0$, we have the following equality, using integration by parts and $\lim_{y\to 0+} y\mu^{(\Lambda,1/y)} = 0$,

$$\frac{\int_0^y \Lambda(dx)}{y\mu^{(\Lambda,1/y)}} = \frac{\int_0^y xx^{-1}\Lambda(dx)}{y\mu^{(\Lambda,1/y)}} = \frac{\int_0^y \mu^{(\Lambda,1/x)}dx - y\mu^{(\Lambda,1/y)}}{y\mu^{(\Lambda,1/y)}}.$$

Together with (20), we have

$$\lim_{y\to 0+} \frac{y\mu^{(\Lambda,1/y)}}{\int_0^y \mu^{(\Lambda,1/x)}dx} = 1.$$

Notice that $\int_0^y \mu^{(\Lambda,1/x)}dx \geq y\mu^{(\Lambda,1/y)}$ and $\mu^{(\Lambda,1/y)}$ is a càglàd function. Hence there exists a càglàd function $f : [0,1] \to [0,1]$, continuous on 0 with $f(0) = 0$ such that

$$\frac{y\mu^{(\Lambda,1/y)}}{\int_0^y \mu^{(\Lambda,1/x)}dx} = 1 - f(y). \tag{21}$$

Notice that the fundamental theorem of Newton and Leibniz works for càglàd functions whose primitive functions take left derivatives. Then we can conclude with (21).

- We now assume that (**) is true. In the first part, we proved implicitly that (20) is equivalent to the right-hand condition of this proposition. Hence we will use (20) to prove (19) which is equivalent to condition (5) and only the first convergence in (19) is needed to be proved. Let M be a positive number and $\frac{M}{n} \leq 1$, $\mu^{(\Lambda,n)} \neq 0$, then

$$\begin{aligned}\frac{\int_{1/n}^{1} x^{-2}\Lambda(dx)}{n\mu^{(\Lambda,n)}} &= \frac{\int_{M/n}^{1} x^{-2}\Lambda(dx)}{n\mu^{(\Lambda,n)}} + \frac{\int_{1/n}^{M/n} x^{-2}\Lambda(dx)}{n\mu^{(\Lambda,n)}} \\ &\leq \frac{1}{M} + 1 - \frac{\mu^{(\Lambda,n/M)}}{\mu^{(\Lambda,n)}}.\end{aligned}$$

The first term is as small as we want by taking M large, and the third term $\frac{\mu^{(\Lambda,n/M)}}{\mu^{(\Lambda,n)}} = exp(-\int_{1/n}^{M/n} \frac{f(x)}{x} ds)\frac{1-f(M/n)}{1-f(1/n)}$. Let $\epsilon > 0$ and n large enough such that $f(x) \leq \epsilon$ on $[1/n, M/n]$. Then $\frac{\mu^{(\Lambda,n/M)}}{\mu^{(\Lambda,n)}} \geq exp(-\epsilon \ln M)(1-\epsilon)$, which is as close as possible to 1 with ϵ small enough. Hence we can conclude.

□

The next corollary is immediate.

Corollary 4.2. *If Λ satisfies (5), then*

- $\lim_{n\to+\infty} \frac{(\mu^{(\Lambda,n)})^k}{n} = 0, \forall k > 0;$
- $\lim_{n\to+\infty} \frac{\mu^{(\Lambda,n)}}{\mu^{(\Lambda,n-M)}} = 1, \forall M > 0;$
- $\lim_{n\to+\infty} \frac{\mu^{(\Lambda,n)}}{\mu^{(\Lambda,n\epsilon)}} = 1, \forall 0 < \epsilon < 1.$

We should next estimate the coalescent process related to the noise measure Λ_1 which serves as a perturbation to the main measure Λ_2. At first, one needs a technical result.

Lemma 4.3. *We assume that $\Lambda(\{0\}) = 0$. Let $g_n^{(\Lambda_1)} = \int_0^1 (1-(1-x)^n - nx(1-x)^{n-1})x^{-2}\Lambda_1(dx)$ in the spirit of (3). Then there exists a positive constant C_2 such that for n large enough*

$$g_n^{(\Lambda_1)} \geq C_2 n^2 \int_0^{1/n} \Lambda_1(dx). \tag{22}$$

Proof. Let $M > 1$. We write

$$\begin{aligned}g_n^{(\Lambda_1)} &= \int_0^1 (1-(1-x)^n - nx(1-x)^{n-1})x^{-2}\Lambda_1(dx) \\ &= \int_0^{\frac{1}{n}} (1-(1-x)^n - nx(1-x)^{n-1})x^{-2}\Lambda_1(dx) \\ &= I_1 + I_2,\end{aligned}$$

where $I_1 = \int_0^{\frac{1}{nM}}(1-(1-x)^n - nx(1-x)^{n-1})x^{-2}\Lambda_1(dx)$ and $I_2 = \int_{\frac{1}{nM}}^{\frac{1}{n}}(1-(1-x)^n - nx(1-x)^{n-1})x^{-2}\Lambda_1(dx)$. It is easy to see that for n large,

$$\begin{aligned} I_1 &\geq \int_0^{\frac{1}{nM}}(n(n-1)-n(n-1)(n-2)x)\frac{1}{2}\Lambda_1(dx) \\ &\geq \int_0^{\frac{1}{nM}}(n(n-1)-(n-1)(n-2)/M)\frac{1}{2}\Lambda_1(dx) \\ &\geq \frac{1}{4}\int_0^{\frac{1}{nM}} n^2\Lambda_1(dx). \end{aligned}$$

For the second term,

$$I_2 \geq \int_{\frac{1}{nM}}^{\frac{1}{n}}(1-(1-\frac{1}{nM})^n - \frac{(1-\frac{1}{nM})^{n-1}}{M})n^2\Lambda_1(dx).$$

Notice that for n large, there exists a positive constant $C(M)$ such that

$$1-(1-\frac{1}{nM})^n - \frac{(1-\frac{1}{nM})^{n-1}}{M} \geq C(M) > 0.$$

Hence $I_2 \geq C(M)\int_{\frac{1}{nM}}^{\frac{1}{n}} n^2\Lambda_1(dx)$. It suffices to take $C_2 = \min\{\frac{1}{4}, C(M)\}$ to conclude. $\square$

The following lemma estimates the coalescent process related to the noise measure Λ_1 when Λ satisfies (5).

Lemma 4.4. *Let Λ satisfy (5). Then for any $M>0$, $0<\epsilon\leq 1$ and n large enough, we have*

$$\mathbb{P}\left(|\Pi^{(\Lambda_1,n)}|(\frac{M}{\mu^{(\Lambda,n)}}) \leq n - n\epsilon\right) = o(n^{-1}). \tag{23}$$

Proof. If $\int_0^{1/n_0}\Lambda(dx) = 0$ with some $n_0>1$, then for any $n>n_0$, $\Lambda_1 = 0$ and hence $|\Pi^{(\Lambda_1,n)}|(t) = n$ for any $t\geq 0$, which proves this lemma. In consequence, one needs only to consider the case where $\int_0^{1/n}\Lambda(dx)\neq 0$ for any $n\geq 1$.

We recall $g_n^{(\Lambda_1)}$ defined in Lemma 4.3. Thanks to Proposition 2.5 where we pick up the notations, $n - \sum_{i=1}^{N(\Lambda_1,n,\frac{M}{\mu^{(\Lambda,n)}})} W_i^{(n)} \leq |\Pi^{(\Lambda_1,n)}|(\frac{M}{\mu^{(\Lambda,n)}})$, where $N(\Lambda_1,n,\frac{M}{\mu^{(\Lambda,n)}})$ is Poisson distributed with parameter $\frac{Mg_n^{(\Lambda_1)}}{\mu^{(\Lambda,n)}}$ independent of $(W_i^{(n)})_{i\geq 1}$ which are i.i.d copies of $X_1^{(\Lambda_1,n)}$. Then we have, for n large,

$$\begin{aligned} \mathbb{P}(|\Pi^{(\Lambda_1,n)}|(\frac{M}{\mu^{(\Lambda,n)}}) \leq n-n\epsilon) &\leq \mathbb{P}(n - \sum_{i=1}^{N(\Lambda_1,n,\frac{M}{\mu^{(\Lambda,n)}})} W_i^{(n)} \leq n-n\epsilon) \\ &= \mathbb{P}(\sum_{i=1}^{N(\Lambda_1,n,\frac{M}{\mu^{(\Lambda,n)}})} W_i^{(n)} - \frac{g_n^{(\Lambda_1)}M}{\mu^{(\Lambda,n)}}\mathbb{E}[W_1^{(n)}] \geq n\epsilon - \frac{g_n^{(\Lambda_1)}M}{\mu^{(\Lambda,n)}}\mathbb{E}[W_1^{(n)}]) \\ &\leq \frac{\mathrm{Var}(\sum_{i=1}^{N(\Lambda_1,n,\frac{M}{\mu^{(\Lambda,n)}})} W_i^{(n)})}{(n\epsilon - \frac{g_n^{(\Lambda_1)}M}{\mu^{(\Lambda,n)}}\mathbb{E}[W_1^{(n)}])^2} = \frac{\frac{Mg_n^{(\Lambda_1)}}{\mu^{(\Lambda,n)}}\mathbb{E}[(W_1^{(n)})^2] - 2(\frac{Mg_n^{(\Lambda_1)}}{2\mu^{(\Lambda,n)}}\mathbb{E}[W_1^{(n)}])^2}{(n\epsilon - \frac{g_n^{(\Lambda_1)}M}{\mu^{(\Lambda,n)}}\mathbb{E}[W_1^{(n)}])^2}, \end{aligned} \tag{24}$$

where the second inequality needs $n\epsilon - \frac{g_n^{(\Lambda_1)}M}{\mu^{(\Lambda,n)}}\mathbb{E}[W_1^{(n)}] > 0$ which is justified by the following calculus: Notice that due to Proposition 2.5 and Lemma 4.3, for n large enough,

$$\mathbb{E}[W_1^{(n)}] + 1 \leq \frac{n(n-1)\int_0^{1/n}\Lambda_1(dx)}{g_n^{(\Lambda_1)}} \leq \frac{1}{C_2}; \mathbb{E}[(W_1^{(n)})^2] \leq \frac{n(n-1)\int_0^{1/n}\Lambda_1(dx)}{g_n^{(\Lambda_1)}} \leq \frac{1}{C_2}, \tag{25}$$

where C_2 is the positive constant in Lemma 4.3.

Notice that (5) gives $\frac{g_n^{(\Lambda_1)}}{n\mu^{(\Lambda,n)}} \leq \frac{g_n^{(\Lambda)}}{n\mu^{(\Lambda,n)}} \to 0$. Then together with (25), we have

$$\frac{g_n^{(\Lambda_1)}M}{\mu^{(\Lambda,n)}}\mathbb{E}[W_1^{(n)}] = o(n), \frac{g_n^{(\Lambda_1)}M}{\mu^{(\Lambda,n)}}\mathbb{E}[(W_1^{(n)})^2] = o(n).$$

Hence $n\epsilon - \frac{g_n^{(\Lambda_1)}M}{\mu^{(\Lambda,n)}}\mathbb{E}[W_1^{(n)}] \asymp n\epsilon$ and (24) gives

$$\mathbb{P}(|\Pi^{(\Lambda_1,n)}|(\frac{M}{\mu^{(\Lambda,n)}}) \leq n - n\epsilon) = o(n^{-1}).$$

Then we conclude for (23). □

The next Lemma helps to explore a property of the main measure Λ_2 when Λ satisfies (5). Notice that $\int_0^1 x^{-2}\Lambda_2 < +\infty$, hence we can use the second coupling construction in subsection 3.1 for $\Pi^{(\Lambda_2,n)}$ and $\tilde{\Pi}^{(\Lambda_2,n)}$. We pick up the notation $P^{(\Lambda_2,n)}$ related to Λ_2 for the probability that singleton $\{1\}$ of $\Pi^{(\Lambda_2,n)}$ not to be coalesced at its *1-st selection time*, as $P^{(\Lambda,n)}$ related to Λ in Remark 3.1.

Lemma 4.5. *Let Λ satisfies (5). Then*

$$\lim_{n\to+\infty} P^{(\Lambda_2,n)} = 0. \tag{26}$$

Proof. Let $(\eta_i^{(\Lambda_2)})_{i\geq 1}$, $(e_i^{(\Lambda_2)})_{i\geq 1}$ be associated to Λ_2 as defined in the second construction in subsection 3.1, and pick up the definition $(\Delta_i^{(\Lambda_2)})_{i\geq 1}$ in Remark 3.1 which shows that

$$P^{(\Lambda_2,n)} \leq \sum_{i=1}^{+\infty}\mathbb{E}[\Delta_i^{(\Lambda_2)}(1-\Delta_i^{(\Lambda_2)})^{n-1}].$$

It is easy to see that $\mathbb{E}[\Delta_i^{(\Lambda_2)}(1-\Delta_i^{(\Lambda_2)})^{n-1}] = \mathbb{E}[\bar{\Delta}_i^{(\Lambda_2)}(1-\bar{\Delta}_i^{(\Lambda_2)})^{n-1}]$, where $\bar{\Delta}_i^{(\Lambda_2)} = \eta_1^{(\Lambda_2)}\Pi_{j=2}^i(1-\eta_j^{(\Lambda_2)})$. It is obvious that $(\bar{\Delta}_i^{(\Lambda_2)})_{i\geq 1}$ is a Markov chain. For $s > 0$, we define a stopping time

$$\begin{aligned}\tau_s &:= \min\{i|\bar{\Delta}_i^{(\Lambda_2)} \leq 1/s\}\\ &= \min\{i| -\sum_{j=2}^{i}\ln(1-\eta_j^{(\Lambda_2)}) \geq \ln s\eta_1^{(\Lambda_2)}\}\\ &= \min\{i+1| -\sum_{j=1}^{i}\ln(1-\eta_{j+1}^{(\Lambda_2)}) \geq \ln s\eta_1^{(\Lambda_2)}\}.\end{aligned}$$

Then we get

$$\begin{aligned}P^{(\Lambda_2,n)} &\leq \mathbb{E}[\sum_{i=1}^{+\infty}\bar{\Delta}_i^{(\Lambda_2)}(1-\bar{\Delta}_i^{(\Lambda_2)})^{n-1}]\\ &= \mathbb{E}[\sum_{i=1}^{\tau_n-1}\bar{\Delta}_i^{(\Lambda_2)}(1-\bar{\Delta}_i^{(\Lambda_2)})^{n-1} + \sum_{i=\tau_n}^{+\infty}\bar{\Delta}_i^{(\Lambda_2)}(1-\bar{\Delta}_i^{(\Lambda_2)})^{n-1}].\end{aligned} \tag{27}$$

Notice that $x(1-x)^{n-1} \leq \frac{1}{n}$, if $\frac{1}{n} \leq x \leq 1$ and $x(1-x)^{n-1} \leq x$, if $0 \leq x \leq \frac{1}{n}$. Then (27) gives

$$P^{(\Lambda_2,n)} \leq \mathbb{E}[\frac{\tau_n - 1}{n} + \sum_{i=\tau_n}^{+\infty}\bar{\Delta}_i^{(\Lambda_2)}] \leq \mathbb{E}[\frac{\tau_n - 1}{n}] + \frac{1}{\mathbb{E}[n\eta_1^{(\Lambda_2)}]}. \tag{28}$$

To calculate $\mathbb{E}[\tau_n]$, we use the renewal theory. Let $\mu = \mathbb{E}[-\ln(1-\eta_1^{(\Lambda_2)})]$.

(1) If $\mu<+\infty$. We denote by $F(t)$ the distribution function of $-\ln(1-\eta_1^{(\Lambda_2)})$ and X an independent random variable with density function $\frac{1}{\mu}(1-F(t))\mathbf{1}_{t\geq 0}$. We define a new Markov chain $(X-\sum_{j=2}^{i}\ln(1-\eta_j^{(\Lambda_2)}))_{i\geq 1}$ and $\tau'_s=\min\{i|X-\sum_{j=1}^{i}\ln(1-\eta_{j+1}^{(\Lambda_2)})\geq \ln s)\}$ for $s>0$. It is clear from the definitions of τ_s and τ'_s that for any $\epsilon\geq 0$

$$\mathbb{E}[\tau'_{s\eta_1^{(\Lambda_2)}}|X=\epsilon]=\mathbb{E}[\tau_{sexp(-\epsilon)}-1].$$

Then

$$\begin{aligned}\mathbb{E}[\tau'_{n\eta_1^{(\Lambda_2)}}]&=\mathbb{E}[\tau'_{n\eta_1^{(\Lambda_2)}}\mathbf{1}_{0\leq X\leq\epsilon}]+\mathbb{E}[\tau'_{n\eta_1^{(\Lambda_2)}}\mathbf{1}_{X>\epsilon}]\\&\geq \mathbb{P}(0\leq X\leq\epsilon)\mathbb{E}[\tau_{nexp(-\epsilon)}-1]+\mathbb{E}[\tau'_{n\eta_1^{(\Lambda_2)}}\mathbf{1}_{X>\epsilon}],\end{aligned}$$

which implies that

$$\mathbb{E}[\tau_{nexp(-\epsilon)}]\leq\frac{\mathbb{E}[\tau'_{n\eta_1^{(\Lambda_2)}}]}{\mathbb{P}(0\leq X\leq\epsilon)}+1. \tag{29}$$

Due to (4.4) and (4.6) in [[14], p.369], we have

$$\mathbb{E}[\tau'_s]=\frac{\ln s}{\mu},\forall s\geq 1.$$

Notice that $\eta_1^{(\Lambda_2)}\geq\frac{1}{n}$, hence $n\eta_1^{(\Lambda_2)}\geq 1$. Therefore, (29) gives

$$\mathbb{E}[\tau_n]\leq\frac{\mathbb{E}[\tau'_{nexp(\epsilon)\eta_1^{(\Lambda_2)}}]}{\mathbb{P}(0\leq X\leq\epsilon)}+1=\frac{\mathbb{E}[\ln(nexp(\epsilon)\eta^{(\Lambda_2)})]}{\mu\mathbb{P}(0\leq X\leq\epsilon)}+1. \tag{30}$$

Notice that for any $0\leq x<1$, we have $-\ln(1-x)\geq x$, hence $\mu\geq\mathbb{E}[\eta_1^{(\Lambda_2)}]=\mu^{(\Lambda,n)}$. Then (30) implies

$$\frac{\mathbb{E}[\tau_n]}{n}\leq\frac{\mathbb{E}[\ln n\eta_1^{(\Lambda_2)}]+\epsilon}{\mathbb{E}[n\eta_1^{(\Lambda_2)}]\mathbb{P}(0\leq X\leq\epsilon)}+\frac{1}{n}. \tag{31}$$

Using (28), it suffices to prove that:

$$\lim_{n\to+\infty}\mathbb{E}[n\eta_1^{(\Lambda_2)}]=+\infty,\text{and}\quad\lim_{n\to+\infty}\frac{\mathbb{E}[\ln(n\eta_1^{(\Lambda_2)})]}{\mathbb{E}[n\eta_1^{(\Lambda_2)}]}=0.$$

It is easy to see that, using (3), there exists a positive constant C_3 such that $\mathbb{E}[n\eta_1^{(\Lambda_2)}]=\frac{n\int_{1/n}^1 x^{-1}\Lambda(dx)}{\int_{1/n}^1 x^{-2}\Lambda(dx)}\geq C_3\frac{n\mu^{(\Lambda,n)}}{g_n^{(\Lambda)}}$, for any $n\geq 3$. Hence $\mathbb{E}[n\eta_1^{(\Lambda_2)}]$ tends to $+\infty$ since Λ satisfies (5). For the second convergence, we fix $M>e$. Then,

$$\begin{aligned}\frac{\mathbb{E}[\ln(n\eta_1^{(\Lambda_2)})]}{\mathbb{E}[n\eta_1^{(\Lambda_2)}]}&=\frac{\mathbb{E}[\ln(n\eta_1^{(\Lambda_2)})\mathbf{1}_{n\eta_1^{(\Lambda_2)}\geq M}]+\mathbb{E}[\ln(n\eta_1^{(\Lambda_2)})\mathbf{1}_{n\eta_1^{(\Lambda_2)}<M}]}{\mathbb{E}[n\eta_1^{(\Lambda_2)}]}\\&\leq\frac{\mathbb{E}[\ln(n\eta_1^{(\Lambda_2)})\mathbf{1}_{n\eta_1^{(\Lambda_2)}\geq M}]}{\mathbb{E}[n\eta_1^{(\Lambda_2)}]}+\frac{\ln M}{\mathbb{E}[n\eta_1^{(\Lambda_2)}]}\\&\leq\frac{\mathbb{E}[\ln(n\eta_1^{(\Lambda_2)})\mathbf{1}_{n\eta_1^{(\Lambda_2)}\geq M}]}{\mathbb{E}[n\eta_1^{(\Lambda_2)}\mathbf{1}_{n\eta_1^{(\Lambda_2)}\geq M}]}+\frac{\ln M}{\mathbb{E}[n\eta_1^{(\Lambda_2)}]}\\&\leq\frac{\ln M}{M}+\frac{\ln M}{\mathbb{E}[n\eta_1^{(\Lambda_2)}]}.\end{aligned}$$

The last inequality is due to the fact that for any $x \geq M > e$, we have $\frac{\ln(x)}{x} \leq \frac{\ln M}{M}$. Since M can be chosen as large as we want, then $\lim_{n\to+\infty} \frac{\mathbb{E}[\ln(n\eta_1^{(\Lambda_2)})]}{\mathbb{E}[n\eta_1^{(\Lambda_2)}]} = 0$. Hence we can conclude.

(2) If $\mu = +\infty$. We define $(\bar{\eta}_i^{(\Lambda_2)})_{i\geq 2} := (\frac{1}{2}\mathbf{1}_{\eta_i^{(\Lambda_2)}\geq\frac{1}{2}} + \eta_i^{(\Lambda_2)}\mathbf{1}_{\eta_i^{(\Lambda_2)}<\frac{1}{2}})_{i\geq 2}$ and for $s > 0$, $\bar{\tau}_s := \min\{i+1|\sum_{j=1}^{i} -\ln(1-\bar{\eta}_{j+1}^{(\Lambda_2)}) \geq \ln s\eta_1^{(\Lambda_2)}\}$. Notice that $\mathbb{E}[-\ln(1-\bar{\eta}_i^{(\Lambda_2)})] < +\infty$, then we return to the first case and get (31) by replacing τ_n by $\bar{\tau}_n$ and keeping the same $\eta_1^{(\Lambda_2)}$ but with different X (depending on $\bar{\eta}_i^{(\Lambda_2)}, i \geq 2$). We see that the closer $\bar{\eta}_i^{(\Lambda_2)}$ is to 1, larger the $-\ln(1-\bar{\eta}_i^{(\Lambda_2)})$ and hence $\tau_n \ll \bar{\tau}_n$. Then we can conclude.

□

The next corollary is straightforward.

Corollary 4.6. *Let Λ satisfies (5) and we define the probability $P^{(\Lambda_2,n,k)}$ related to $\Pi^{(\Lambda_2,n)}$ for at least one among the first k singletons being not coalesced at their 1-st selection times. Then*

$$P^{(\Lambda_2,n,k)} \leq kP^{(\Lambda_2,n)}, \tag{32}$$

and $\lim_{n\to+\infty} P^{(\Lambda_2,n,k)} = 0$.

Before proving Theorem 1.2, we first give a trivial lemma whose proof is omitted.

Lemma 4.7. *Let Π be a realization of $\Pi^{(\Lambda_1,n)}$ and $T_1^{(\Lambda,n)}$ be the external branch length of $\{1\}$ after the addition of Λ_2 to Π using measure division construction. Assume that for a time $t > 0$, there exists $2 \leq m \leq n$, such that $m \leq |\Pi|(t) \leq n$ and $\{1\} \notin \Pi(t)$. Then after the adding of Λ_2 to Π using measure division construction, for any $0 \leq t' \leq t$, we have*

$$\mathbb{P}(T_1^{(\Lambda_2,m)} \leq t') \leq \mathbb{P}(T_1^{(\Lambda,n)} \leq t'|\Pi^{(\Lambda_1,n)} = \Pi) \leq \mathbb{P}(T_1^{(\Lambda_2,n)} \leq t').$$

Proof of Theorem 1.2

Proof. Since the case $\int_0^1 x^{-1}\Lambda(dx) < +\infty$ has been proved in [23], we assume that $\int_0^1 x^{-1}\Lambda(dx) = +\infty$. The theorem is equivalent to say that

$$\lim_{n\to+\infty} \mathbb{P}(T_1^{(\Lambda,n)} \geq \frac{t}{\mu^{(\Lambda,n)}}) = exp(-t), \forall t \geq 0. \tag{33}$$

Let $T_1^{(\Lambda,n)}$ be obtained from the measure division constructed $\Pi_{1,2}^{(\Lambda,n)}$ with noise measure $\Lambda_1$and main measure Λ_2. Let $0 < \epsilon < 1$. Notice that under the event $\{|\Pi^{(\Lambda_1,n)}|(\frac{t}{\mu^{(\Lambda,n)}}) \geq n-n\epsilon\}$, the probability for singleton $\{1\}$ to be coalesced at time $\frac{t}{\mu^{(\Lambda,n)}}$ is less than or equal to $\frac{\lceil 2n\epsilon\rceil}{n}$ using exchangeability property (since there are at most $\lceil 2n\epsilon\rceil$ singletons merged). We denote by $\kappa_n(t) := \mathbb{P}(|\Pi^{(\Lambda_1,n)}|(\frac{t}{\mu^{(\Lambda,n)}}) < n-n\epsilon)$ and $\kappa_n(t) = o(n^{-1})$ due to inequality (23). Then for n large enough, we have $\mathbb{P}(E') \geq (1-\kappa_n(t))(1-\frac{\lceil 2n\epsilon\rceil}{n})$, where

$$E' := \{|\Pi^{(\Lambda_1,n)}|(\frac{t}{\mu^{(\Lambda,n)}}) \geq n-n\epsilon\}\bigcap\{T_1^{(\Lambda_1,n)} > \frac{t}{\mu^{(\Lambda,n)}}\}.$$

Lemma 4.7 gives that

$$\mathbb{P}(T_1^{(\Lambda_2,\lfloor n-n\epsilon\rfloor)} \geq \frac{t}{\mu^{(\Lambda,n)}}) \geq \mathbb{P}(T_1^{(\Lambda,n)} \geq \frac{t}{\mu^{(\Lambda,n)}}|E') \geq \mathbb{P}(T_1^{(\Lambda_2,n)} \geq \frac{t}{\mu^{(\Lambda,n)}}). \tag{34}$$

We prove at first $\lim_{n\to+\infty} \mathbb{P}(T_1^{(\Lambda_2,n)} \geq \frac{t}{\mu^{(\Lambda,n)}}) = exp(-t)$. Recall that $L_1^{(\Lambda_2,n,1)}$ is the *1-st selection time* of $\{1\}$. The second construction in subsection 3.1 gives that

$$\mathbb{P}(L_1^{(\Lambda_2,n,1)} \geq \frac{t}{\mu^{(\Lambda,n)}}) + P^{(\Lambda_2,n)} \geq \mathbb{P}(T_1^{(\Lambda_2,n)} \geq \frac{t}{\mu^{(\Lambda,n)}}) \geq \mathbb{P}(L_1^{(\Lambda_2,n,1)} \geq \frac{t}{\mu^{(\Lambda,n)}}). \tag{35}$$

It is revealed in the same construction that $\mathbb{P}(L_1^{(\Lambda_2,n,1)} \geq \frac{t}{\mu^{(\Lambda,n)}}) = exp(-t)$. Hence using Lemma 4.5, we get

$$\lim_{n\to+\infty} \mathbb{P}(T_1^{(\Lambda_2,n)} \geq \frac{t}{\mu^{(\Lambda,n)}}) = exp(-t).$$

We obtain also $\lim_{n\to+\infty} \mathbb{P}(T_1^{(\Lambda_2,\lfloor n-n\epsilon\rfloor)} \geq \frac{t}{\mu^{(\Lambda,n)}}) = exp(-t)$, using $\lim_{n\to+\infty} \frac{\mu^{(\Lambda,n-n\epsilon)}}{\mu^{(\Lambda,n)}} = 1$ which is due to Corollary 4.2.

Then (34) gives that $\lim_{n\to+\infty} \mathbb{P}(T_1^{(\Lambda,n)} \geq \frac{t}{\mu^{(\Lambda,n)}}|E') = exp(-t)$. Since $\mathbb{P}(E')$ is as close as we want to 1 by taking ϵ small and n large, we get $\lim_{n\to+\infty} \mathbb{P}(T_1^{(\Lambda,n)} \geq \frac{t}{\mu^{(\Lambda,n)}}) = exp(-t)$, which allows to conclude. □

Proof of Theorem 1.3

Proof. We prove instead for $k \in \mathbb{N}$:

$$\mu^{(\Lambda,n)}(T_1^{(\Lambda,n)}, T_2^{(\Lambda,n)}, \cdots, T_k^{(\Lambda,n)}) \xrightarrow{(d)} (e_1, e_2, \cdots, e_k), \tag{36}$$

which is equivalent to (7) (see Billingsley [[3], p.19]). We will give the proof for $k = 2$ and leave the easy extension to readers. We have to prove:

$$\lim_{n\to+\infty} \mathbb{P}(T_1^{(\Lambda,n)} \geq \frac{t_1}{\mu^{(\Lambda,n)}}, T_2^{(\Lambda,n)} \geq \frac{t_2}{\mu^{(\Lambda,n)}}) = exp(-t_1 - t_2), \forall 0 \leq t_1 \leq t_2. \tag{37}$$

Similar to the event E' defined in the proof of Theorem 1.2, we define another event, with $0 < \epsilon < 1$

$$E'' := \{|\Pi^{(\Lambda_1,n)}|(\frac{t_2}{\mu^{(\Lambda,n)}}) \geq n - n\epsilon\} \bigcap \{T_1^{(\Lambda_1,n)} \geq \frac{t_1}{\mu^{(\Lambda,n)}}, T_2^{(\Lambda_1,n)} \geq \frac{t_2}{\mu^{(\Lambda,n)}}, \}.$$

Recall that $\{|\Pi^{(\Lambda_1,n)}|(\frac{t_2}{\mu^{(\Lambda,n)}}) \geq n - n\epsilon\}$ implies that there are at least $n - \lceil 2n\epsilon\rceil$ singletons at time $\frac{t_2}{\mu^{(\Lambda,n)}}$. For n large enough, using exchangeability, we have $\mathbb{P}(E'') \geq \frac{\binom{n-\lceil 2n\epsilon\rceil}{2}}{\binom{n}{2}}(1 - \kappa_n(t_2))$, where $\kappa_n(t_2) = \mathbb{P}(|\Pi^{(\Lambda_1,n)}|(\frac{t_2}{\mu^{(\Lambda,n)}}) < n - n\epsilon)$. For ϵ small enough and n large enough, we have $\mathbb{P}(E'')$ as close as we want to 1.

Similar to (34), using the event E'' and also $\lim_{n\to+\infty} \frac{\mu^{(\Lambda,n-n\epsilon)}}{\mu^{(\Lambda,n)}} = 1$, we need only to get the following convergence to prove (37):

$$\lim_{n\to+\infty} \mathbb{P}(T_1^{(\Lambda_2,n)} \geq \frac{t_1}{\mu^{(\Lambda,n)}}, T_2^{(\Lambda_2,n)} \geq \frac{t_2}{\mu^{(\Lambda,n)}}) = exp(-t_1 - t_2). \tag{38}$$

Using the second construction in subsection 3.1, due to the same arguments to get (35), we have

$$\begin{aligned}
&\mathbb{P}(L_1^{(\Lambda_2,n,1)} \geq \frac{t_1}{\mu^{(\Lambda,n)}}, L_2^{(\Lambda_2,n,1)} \geq \frac{t_2}{\mu^{(\Lambda,n)}}) + P^{(\Lambda_2,n,2)} \\
&\qquad \geq \mathbb{P}(T_1^{(\Lambda_2,n)} \geq \frac{t_1}{\mu^{(\Lambda,n)}}, T_2^{(\Lambda_2,n)} \geq \frac{t_2}{\mu^{(\Lambda,n)}}) \geq \mathbb{P}(L_1^{(\Lambda_2,n,1)} \geq \frac{t_1}{\mu^{(\Lambda,n)}}, L_2^{(\Lambda_2,n,1)} \geq \frac{t_2}{\mu^{(\Lambda,n)}}).
\end{aligned}$$

Using Corollary 4.6, it suffices to obtain the following convergence to conclude:

$$\lim_{n\to+\infty} \mathbb{P}(L_1^{(\Lambda_2,n,1)} \geq \frac{t_1}{\mu^{(\Lambda,n)}}, L_2^{(\Lambda_2,n,1)} \geq \frac{t_2}{\mu^{(\Lambda,n)}}) = exp(-t_1 - t_2). \tag{39}$$

Indeed, adopting the notations in the second construction in subsection 3.1, within $[0, \frac{t_1}{\mu^{(\Lambda,n)}}]$, there are N_1 *marking times* and for $(\frac{t_1}{\mu^{(\Lambda,n)}}, \frac{t_2}{\mu^{(\Lambda,n)}}]$, there are N_2 *marking times*. N_1 and N_2 are independently Poisson distributed with parameter respectively $\frac{t_1 \int_{1/n}^1 x^{-2}\Lambda(dx)}{\mu^{(\Lambda,n)}}$ and $\frac{(t_2-t_1)\int_{1/n}^1 x^{-2}\Lambda(dx)}{\mu^{(\Lambda,n)}}$. Then we get

$$\begin{aligned}
&\mathbb{P}(L_1^{(\Lambda_2,n,1)} \geq \frac{t_1}{\mu^{(\Lambda,n)}}, L_2^{(\Lambda_2,n,1)} \geq \frac{t_2}{\mu^{(\Lambda,n)}}) \\
&= \mathbb{E}[\Pi_{i=1}^{N_1}(1-\eta_i^{(\Lambda_2)})^2 \Pi_{i=N_1+1}^{N_1+N_2}(1-\eta_i^{(\Lambda_2)})] \\
&= \mathbb{E}[(1-2\mathbb{E}[\eta_1^{(\Lambda_2)}]+\mathbb{E}[(\eta_1^{(\Lambda_2)})^2])^{N_1}]\mathbb{E}[(1-\mathbb{E}[\eta_1^{(\Lambda_2)}])^{N_2}] \\
&= exp\left(\frac{\int_{1/n}^1 x^{-2}\Lambda(dx)t_1}{\mu^{(\Lambda,n)}}(e^{1-2\mathbb{E}[\eta_1^{(\Lambda_2)}]+\mathbb{E}[(\eta_1^{(\Lambda_2)})^2]}-1)\right) exp\left(\frac{\int_{1/n}^1 x^{-2}\Lambda(dx)(t_2-t_1)}{\mu^{(\Lambda,n)}}(e^{1-\mathbb{E}[\eta_1^{(\Lambda_2)}]}-1)\right),
\end{aligned}$$

where the last equality is due to the moment generating function of Poisson distribution. Recall that $\mathbb{E}[\eta_1^{(\Lambda_2)}] = \frac{\int_{1/n}^1 x^{-1}\Lambda(dx)}{\int_{1/n}^1 x^{-2}\Lambda(dx)}$ and $\mathbb{E}[(\eta_1^{(\Lambda_2)})^2] = \frac{\int_{1/n}^1 \Lambda(dx)}{\int_{1/n}^1 x^{-2}\Lambda(dx)}$. Then

$$\frac{\int_{1/n}^1 x^{-2}\Lambda(dx)t_1}{\mu^{(\Lambda,n)}}\mathbb{E}[\eta_1^{(\Lambda_2)}] = \frac{\int_{1/n}^1 x^{-1}\Lambda(dx)}{\mu^{(\Lambda,n)}} = 1, \text{and} \quad \frac{\int_{1/n}^1 x^{-2}\Lambda(dx)t_1}{\mu^{(\Lambda,n)}}\mathbb{E}[(\eta_1^{(\Lambda_2)})^2] = \frac{\int_{1/n}^1 \Lambda(dx)}{\mu^{(\Lambda,n)}} \to 0 \ .$$

Then we conclude for (39). □

Proof of Corollary 1.4

Proof. We prove at first the case with one external branch length. Lemma 5.7 in [11] allows to just prove that for any $k \in \mathbb{N}$, $\sup\{\mathbb{E}[(\mu^{(\Lambda,n)}T_1^{(\Lambda,n)})^k]|n \geq 2\} < +\infty$. Let $M > 0, 0 < \epsilon < 1$, $\beta_n = |\Pi^{(\Lambda,n)}|(\frac{M}{\mu^{(\Lambda,n)}})$ and $n_0 := \min\{i|\mu^{(i)} > 0\}$. To avoid invalid calculus, we set $\mu^{(\Lambda,n)} = 1$ if $n < n_0$. Using Markov property, we have

$$T_1^{(\Lambda,n)} \ll \frac{M}{\mu^{(\Lambda,n)}} + \bar{T}_1^{(\Lambda,\beta_n)}\mathbf{1}_{T_1^{(\Lambda,n)} \geq \frac{M}{\mu^{(\Lambda,n)}}},$$

where $\bar{T}_1^{(\Lambda,n)} \stackrel{(d)}{=} T_1^{(\Lambda,n)}, n \geq 2$ and $\bar{T}_1^{(\Lambda,n)}$ is independent of everything. Then for $n\epsilon \geq n_0$,

$$\begin{aligned}
\mathbb{E}[(\mu^{(\Lambda,n)}T_1^{(\Lambda,n)})^k] &\leq \mathbb{E}[(M+\mu^{(\Lambda,n)}\bar{T}_1^{(\Lambda,\beta_n)}\mathbf{1}_{\mu^{(\Lambda,n)}T_1^{(\Lambda,n)}>M})^k] \leq (2M)^k + \mathbb{E}[(2\mu^{(\Lambda,n)}\bar{T}_1^{(\Lambda,\beta_n)}\mathbf{1}_{\mu^{(\Lambda,n)}T_1^{(\Lambda,n)}>M})^k] \\
&\leq (2M)^k + (\mathbb{E}[2\mu^{(\Lambda,n)}\bar{T}_1^{(n)}\mathbf{1}_{\beta_n=n}])^k] + \mathbb{E}[(2\mu^{(\Lambda,n)}\bar{T}_1^{(\Lambda,\beta_n)}\mathbf{1}_{\mu^{(\Lambda,n)}T_1^{(\Lambda,n)}>M, n\epsilon\leq\beta_n\leq n-1})^k] \\
&\quad + \mathbb{E}[(2\mu^{(\Lambda,n)}\bar{T}_1^{(\Lambda,\beta_n)}\mathbf{1}_{\mu^{(\Lambda,n)}T_1^{(\Lambda,n)}>M,\beta_n<n\epsilon})^k] \\
&\leq (2M)^k + exp(-\frac{Mg_n^{(\Lambda)}}{\mu^{(\Lambda,n)}})\mathbb{E}[(2\mu^{(\Lambda,n)}\bar{T}_1^{(n)})^k] \\
&\quad + \mathbb{P}(\mu^{(\Lambda,n)}T_1^{(\Lambda,n)} > M)(2\frac{\mu^{(\Lambda,n)}}{\mu^{(\Lambda,n\epsilon)}})^k \max\{\mathbb{E}[(\mu^{(\Lambda,j)}\bar{T}_1^{(\Lambda,j)})^k]|j \in [n\epsilon, n-1]\} \\
&\quad + \mathbb{P}(\beta_n < n\epsilon)\mathbb{E}[\frac{\beta_n}{n}(2\frac{\mu^{(\Lambda,n)}}{\mu^{(\Lambda,\beta_n)}})^k(\mu^{(\Lambda,\beta_n)}\bar{T}_1^{(\Lambda,\beta_n)})^k|\beta_n < n\epsilon],
\end{aligned} \tag{40}$$

where $exp(-\frac{Mg_n^{(\Lambda)}}{\mu^{(\Lambda,n)}})$ in the second term at right of the last inequality is the probability for no coalescence within $[0, \frac{M}{\mu^{(\Lambda,n)}}]$. The third term at right of the last inequality is due to the increasing property of $\mu^{(\Lambda,n)}$ on n when $n \geq n_0$. The fourth term is due to exchangeability which says that the probability for $\{1\}$ not to be coalescent at time $\frac{M}{\mu^{(\Lambda,n)}}$ when there exist only β_n blocks is less than $\frac{\beta_n}{n}$. One needs the following three estimations to prove the boundedness of $(\mathbb{E}[(\mu^{(\Lambda,n)}T_1^{(\Lambda,n)})^k])_{n\geq 2}$.

- Estimation of $exp(-\frac{Mg_n^{(\Lambda)}}{\mu^{(\Lambda,n)}})$: Notice that for $n \geq n_0$,

$$\frac{g_n^{(\Lambda)}}{\mu^{(\Lambda,n)}} = \frac{\int_0^1(1-(1-x)^n - nx(1-x)^{n-1})x^{-2}\Lambda(dx)}{\int_{1/n}^1 x^{-1}\Lambda(dx)} \geq \frac{\int_{1/n}^1(1-(1-x)^n - nx(1-x)^{n-1})x^{-2}\Lambda(dx)}{\int_{1/n}^1 x^{-1}\Lambda(dx)} \geq \frac{e-2}{e}.$$

And if $2 \leq n < n_0$, we have $exp(-\frac{Mg_n^{(\Lambda)}}{\mu^{(\Lambda,n)}}) = exp(-Mg_n^{(\Lambda)}) \overset{M\to+\infty}{\to} 0$. Hence if M large, we have, for any $n \geq 2$,

$$exp(-\frac{Mg_n^{(\Lambda)}}{\mu^{(\Lambda,n)}}) \leq \frac{1}{4}. \tag{41}$$

- Estimation of $\mathbb{P}(\mu^{(\Lambda,n)}T_1^{(\Lambda,n)} > M)(2\frac{\mu^{(\Lambda,n)}}{\mu^{(\Lambda,n\epsilon)}})^k$: Due to Corollary 4.2, we get $\lim_{n\to+\infty}\frac{\mu^{(\Lambda,n)}}{\mu^{(\Lambda,n\epsilon)}} = 1$, and Theorem 1.2 gives $\lim_{n\to+\infty}\mathbb{P}(\mu^{(\Lambda,n)}T_1^{(\Lambda,n)} > M) = exp(-M)$. Hence by taking M large, we have for any $n \geq 2$,

$$\mathbb{P}(\mu^{(\Lambda,n)}T_1^{(\Lambda,n)} > M)(2\frac{\mu^{(\Lambda,n)}}{\mu^{(\Lambda,n\epsilon)}})^k \leq \frac{1}{2}. \tag{42}$$

- Estimation of $\frac{\beta_n}{n}(2\frac{\mu^{(\Lambda,n)}}{\mu^{(\Lambda,\beta_n)}})^k, \beta_n < n\epsilon$: Using the notations in Proposition 4.1, for $\beta_n \geq n_0$, we have

$$\frac{\mu^{(\Lambda,n)}}{\mu^{(\Lambda,\beta_n)}} = exp(\int_{1/n}^{1/\beta_n}\frac{f(x)}{x}dx)\frac{1-f(1/n)}{1-f(1/\beta_n)}. \tag{43}$$

Let $n_1 > n_0$ such that for any $n \geq n_1$, we have $f(1/n) \leq \frac{1}{2k}$. Hence for any $a, b \geq n_1$, $\frac{1-f(a)}{1-f(b)} \leq 2$. This n_1 can be found since $f(1/n)$ tends to 0 as n tends to $+\infty$. Then (43) implies, for $\beta_n \geq n_1$,

$$\frac{\mu^{(\Lambda,n)}}{\mu^{(\Lambda,\beta_n)}} \leq 2(\frac{n}{\beta_n})^{\frac{1}{2k}}.$$

Hence if $n_1 \leq \beta_n < n\epsilon$ and $\epsilon \leq 4^{-2k-1}$,

$$\frac{\beta_n}{n}(2\frac{\mu^{(\Lambda,n)}}{\mu^{(\Lambda,\beta_n)}})^k \leq 4^k(\frac{\beta_n}{n})^{1/2} < 4^k(\epsilon)^{1/2} \leq \frac{1}{2}.$$

If $\beta_n < n_1$, due to Corollary 4.2, when n large enough, we have

$$\frac{\beta_n}{n}(2\frac{\mu^{(\Lambda,n)}}{\mu^{(\Lambda,\beta_n)}})^k \leq \frac{1}{2}.$$

In total, when n is large enough and $\beta_n < n\epsilon$, then

$$\frac{\beta_n}{n}(2\frac{\mu^{(\Lambda,n)}}{\mu^{(\Lambda,\beta_n)}})^k \leq \frac{1}{2}. \tag{44}$$

Using (40), (41), (42) and (44), we get

$$\begin{aligned}\mathbb{E}[(\mu^{(\Lambda,n)}T_1^{(\Lambda,n)})^k] &\leq \frac{4}{3}(2M)^k + \frac{2}{3}\max\{\mathbb{E}[(\mu^{(\Lambda,j)}\bar{T}_1^{(\Lambda,j)})^k]|j \in [n\epsilon, n-1]\} + \frac{2}{3}\mathbb{E}[\mu^{(\Lambda,\beta_n)}\bar{T}_1^{(\Lambda,\beta_n)})^k|\beta_n < n\epsilon] \\ &\leq \frac{4}{3}(2M)^k + \frac{2}{3}\max\{\mathbb{E}[(\mu^{(\Lambda,j)}\bar{T}_1^{(\Lambda,j)})^k]|j \leq n-1\}. \qquad (45)\end{aligned}$$

The above inequality is valid for a large M, $\epsilon = 4^{-2k-1}$ and $n \geq n_1$. Let $C_4 \geq \max\{\mathbb{E}[(\mu^{(\Lambda,j)}T_1^{(\Lambda,j)})^k], 4(2M)^k|2 \leq j < n_1\}$, then for any $n \geq 2$, $C_4 \geq \mathbb{E}[(\mu^{(\Lambda,n)}T_1^{(\Lambda,n)})^k]$ using (45). Then we can conclude.

The case of multiple external branch lengths is merely a consequence of the case with one external branch and also of Lemma 5.7 in [11], Theorem 1.3 and Cauchy-Schwarz inequality. □

Proof of Corollary 1.5

Proof. Corollary 1.4 shows that $\lim_{n\to+\infty} \mathbb{E}[\mu^{(\Lambda,n)} L_{ext}^{(\Lambda,n)}/n] = 1$ and $\lim_{n\to+\infty} \mathrm{Var}(\mu^{(\Lambda,n)} L_{ext}^{(\Lambda,n)}/n) = 0$, hence $\mu^{(\Lambda,n)} L_{ext}^{(\Lambda,n)}/n$ converges in L^2 to 1 which deduce the convergence in probability. □

Before proving Corollary 1.6, we study at first a problem of sensibility of a recurrence satisfied by $(T_1^{(\Lambda,n)})_{n\geq 2}$. More precisely, if $a_n = \mathbb{E}[T_1^{(\Lambda,n)}]$, then a_n satisfies a recurrence (see [9]): $a_1 = 0$, and for $n \geq 2$, we have

$$a_n = c_n + \sum_{k=1}^{n-1} p_{n,k} \frac{k-1}{n} a_k, \tag{46}$$

where $(c_n)_{n\geq 2} = (\frac{1}{g_n^{(\Lambda)}})_{n\geq 2}$ and $p_{n,k} = p_{n,k}^{(\Lambda)}$. Due to Theorem 1.4, we have $\lim_{n\to+\infty} \mu^{(\Lambda,n)} a_n = 1$. The question is as follows: what is the limit behavior of a_n if we set initially the values of $(a_i)_{1\leq i\leq n_0}$ with $n_0 \geq 1$ without using (46) and replace c_n by $c'_n = \frac{1}{g_n^{(\Lambda)}} + o(\frac{1}{g_n^{(\Lambda)}})$? It is answered in the next lemma.

Lemma 4.8. *Let $(a'_i)_{1\leq i\leq n_0}$ be n_0 real numbers and for $n > n_0$*

$$a'_n = c'_n + \sum_{k=1}^{n-1} p_{n,k} \frac{k-1}{n} a'_k, \tag{47}$$

where $(c'_n)_{n>n_0}$ is a sequence which satisfies $c'_n = \frac{1}{g_n^{(\Lambda)}} + o(\frac{1}{g_n^{(\Lambda)}})$. Then

$$\lim_{n\to+\infty} \mu^{(\Lambda,n)} a'_n = 1.$$

Proof. We fix $\epsilon > 0$ and let $n_\epsilon > n_0$ such that $c'_n \leq \frac{1+\epsilon}{g_n^{(\Lambda)}}$ for $n > n_\epsilon$. We set $M = \max\{|a'_i|, a_i | 1 \leq i \leq n_\epsilon\}$.

Let us firstly look at (46) which has the following interpretation using random walk: A walker stands initially at point n, then after time c_n, he jumps to point k_1 with probability p_{n,k_1}, then after time $\frac{k_1-1}{n} c_{k_1}$, he jumps to k_2 with probability p_{k_1,k_2}, and then after time $\frac{(k_1-1)(k_2-1)}{nk_1} c_{k_2}$, he jumps to the next point, etc. If he falls at point 1, then this walk is finished. It is easy to see that a_n is the expectation of the total walking time.

The recurrence (47) has the same interpretation. The difference is that one should stop the walker when he arrives at a point i within $[1, n_0]$ and one adds a'_i to the walk time (notice that a'_i can be non-positive). To estimate a'_n, we use a Markov chain $(W_i)_{i\geq 0}$ to describe the jumping structure of (46) and (47) : $W_0 = n$,

- If $W_i = k$ with $k \geq n_\epsilon$, then $W_{i+1} = k'$ with probability $p_{k,k'}$, where $1 \leq k' \leq k-1$;
- If $W_i < n_\epsilon$, then we set $W_j = W_i$ for any $j \geq i+1$.

Notice that the jumping dynamics of both recurrences is characterized by $(W_i)_{i\geq 0}$ until arriving at a point within $[1, n_\epsilon]$. And also we see that $(W_i)_{i\geq 0}$ is the block counting process $|\Pi^{(\Lambda,n)}|$ stopped when the first time arriving within $[1, n_\epsilon]$.

Let $\varsigma_n = \min\{i | W_i = W_{i+1}\}$, $C_{\varsigma_n} = \Pi_{i=0}^{\varsigma_n - 1} \frac{W_{i+1}-1}{W_i}$ and T_{ς_n} is set to be the time to ς_n of the random walk related to (46) and T'_{ς_n} being that of (47).

It is easy to see the following relations:

$$a_n - M\mathbb{E}[C_{\varsigma_n}] \leq \mathbb{E}[T_{\varsigma_n}] \leq a_n; \quad a'_n - M\mathbb{E}[C_{\varsigma_n}] \leq \mathbb{E}[T'_{\varsigma_n}] \leq a'_n + M\mathbb{E}[C_{\varsigma_n}]; \quad \mathbb{E}[T'_{\varsigma_n}] \leq (1+\epsilon)\mathbb{E}[T_{\varsigma_n}]$$

Notice that $\mathbb{E}[C_{\varsigma_n}] \leq \frac{n_\epsilon}{n}$ and due to Corollary 4.2, we have $\lim_{n\to+\infty} \frac{M\mu^{(\Lambda,n)}}{n} = 0$. Hence $\lim_{n\to+\infty} M\mathbb{E}[C_{\varsigma_n}]\mu^{(\Lambda,n)} = 0$. Then we can conclude that for n large, $a'_n \leq (1+2\epsilon)a_n$. In the same way, we can prove also $a'_n \geq (1-2\epsilon')a_n$ for another small positive number ϵ' with n large enough. Hence we deduce the lemma. □

Proof of Corollary 1.6

Proof. Let $b_n = \mathbb{E}[\mu^{(\Lambda,n)} L^{(\Lambda,n)}/n]$. Then looking at the first coalescence of the process $\Pi^{(\Lambda,n)}$, we have,

$$b_1 = 0; b_n = \frac{\mu^{(\Lambda,n)}}{g_n^{(\Lambda)}} + \sum_{k=1}^{n-1} p_{n,k} \frac{k\mu^{(\Lambda,n)}}{n\mu^{(\Lambda,k)}} b_k, n \geq 2. \tag{48}$$

If for some k, $\mu^{(\Lambda,k)} = 0$, then we set $\mu^{(\Lambda,k)} = 1$. To use Lemma 4.8, we write (48) as:

$$b_1 = 0; b_n = \frac{\mu^{(\Lambda,n)}}{g_n^{(\Lambda)}} + \sum_{k=1}^{n-1} p_{n,k} \frac{\mu^{(\Lambda,n)}}{n\mu^{(\Lambda,k)}} b_k + \sum_{k=1}^{n-1} p_{n,k} \frac{(k-1)\mu^{(\Lambda,n)}}{n\mu^{(\Lambda,k)}} b_k, n \geq 2. \tag{49}$$

We prove that $\sum_{k=1}^{n-1} p_{n,k} \frac{\mu^{(\Lambda,n)}}{n\mu^{(\Lambda,k)}} = o(\frac{\mu^{(\Lambda,n)}}{g_n^{(\Lambda)}})$. Indeed, due to (9), let $a = \int_0^1 (1-(1-x)^{n-1}) x^{-1} \Lambda(dx)$ and $M > 0$, then

$$\mathbb{P}(X_1^{(\Lambda,n)} \geq Ma) \leq \frac{\mathbb{E}[X_1^{(\Lambda,n)}]}{Ma} \leq \frac{n}{Mg_n^{(\Lambda)}}. \tag{50}$$

Using Corollary 4.2, we have $\limsup_{n\to+\infty} \frac{a}{n} \leq \lim_{n\to+\infty} \frac{\int_0^{1/n} (n-1)\Lambda(dx) + \mu^{(\Lambda,n)}}{n} = 0$, $\lim_{n\to+\infty} \frac{\mu^{(\Lambda,n)}}{\mu^{(\Lambda,n-Ma)}} = 1$. Then for n large enough,

$$\begin{aligned}
\sum_{k=1}^{n-1} p_{n,k} \frac{\mu^{(\Lambda,n)}}{n\mu^{(\Lambda,k)}} &= \sum_{k=1}^{\lfloor n-Ma \rfloor} p_{n,k} \frac{\mu^{(\Lambda,n)}}{n\mu^{(\Lambda,k)}} + \sum_{k=\lfloor n-Ma \rfloor+1}^{n-1} p_{n,k} \frac{\mu^{(\Lambda,n)}}{n\mu^{(\Lambda,k)}} \\
&\leq \mathbb{P}(X_1^{(\Lambda,n)} \geq Ma) \mathbb{E}[\frac{\mu^{(\Lambda,n)}}{n\mu^{(\Lambda,n-X_1^{(\Lambda,n)})}} | X_1^{(\Lambda,n)} \geq Ma] + \frac{\mu^{(\Lambda,n)}}{\mu^{(\Lambda,n-Ma)} n} \\
&\leq \frac{\mu^{(\Lambda,n)}}{Mg_n^{(\Lambda)}} \max\{\frac{1}{\mu^{(\Lambda,k)}} | 1 \leq k \leq n\} + \frac{\mu^{(\Lambda,n)}}{\mu^{(\Lambda,n-Ma)} n},
\end{aligned}$$

where the first term at right of the the last inequality is due to (50) and can be as small as we want w.r.t $\frac{\mu^{(\Lambda,n)}}{g_n^{(\Lambda)}}$ when M is large enough. Notice that $n^{-1} = o(\frac{\mu^{(\Lambda,n)}}{g_n^{(\Lambda)}})$ due to (5). Then the second term $\frac{\mu^{(\Lambda,n)}}{\mu^{(\Lambda,n-Ma)} n} = o(\frac{\mu^{(\Lambda,n)}}{g_n^{(\Lambda)}})$ using also $\lim_{n\to+\infty} \frac{\mu^{(\Lambda,n)}}{\mu^{(\Lambda,n-Ma)}} = 1$. Then $\sum_{k=1}^{n-1} p_{n,k} \frac{\mu^{(\Lambda,n)}}{n\mu^{(\Lambda,k)}} = o(\frac{\mu^{(\Lambda,n)}}{g_n^{(\Lambda)}})$.

In fact, it suffices to prove that $(b_k)_{k\geq 2}$ are bounded, since in this case, $\sum_{k=1}^{n-1} p_{n,k} \frac{\mu^{(\Lambda,n)}}{n\mu^{(\Lambda,k)}} b_k = o(\frac{\mu^{(\Lambda,n)}}{g_n^{(\Lambda)}})$ and we apply Lemma 4.8 to (49).

To control $\sum_{k=1}^{n-1} p_{n,k} \frac{\mu^{(\Lambda,n)}}{n\mu^{(\Lambda,k)}} b_k$, we construct another recurrence:

$$b'_1 = 0; b'_n = \frac{C\mu^{(\Lambda,n)}}{g_n^{(\Lambda)}} + \sum_{k=1}^{n-1} p_{n,k} \frac{(k-1)\mu^{(\Lambda,n)}}{n\mu^{(\Lambda,k)}} b'_k, n \geq 2. \tag{51}$$

where C is a positive number. If $C = 1$, this is exactly a transformation of the recurrence (46). Let $M'(C) = \sup\{b'_n\}$. Then it is easy to see that $M'(C) = CM'(1)$. Let $n_0 \geq 1$, such that for $n \geq n_0$, we have $\sum_{k=1}^{n-1} p_{n,k} \frac{\mu^{(\Lambda,n)}}{n\mu^{(\Lambda,k)}} M'(1) \leq \frac{1}{2} \frac{\mu^{(\Lambda,n)}}{g_n^{(\Lambda)}}$. Then for $C \geq 2, n \geq n_0$,

$$\frac{\mu^{(\Lambda,n)}}{g_n^{(\Lambda)}} + \sum_{k=1}^{n-1} p_{n,k} \frac{\mu^{(\Lambda,n)}}{n\mu^{(\Lambda,k)}} M'(C) \leq \frac{C\mu^{(\Lambda,n)}}{g_n^{(\Lambda)}}. \tag{52}$$

For $2 \leq n < n_0$, we set C large enough such that

$$\frac{\mu^{(\Lambda,n)}}{g_n^{(\Lambda)}} + \sum_{k=1}^{n-1} p_{n,k} \frac{\mu^{(\Lambda,n)}}{n\mu^{(\Lambda,k)}} \max\{b_i | 1 \le i < n_0\} \le \frac{C\mu^{(\Lambda,n)}}{g_n^{(\Lambda)}}. \tag{53}$$

Comparing the coefficients and initial values of recurrences (49) and (51) using (52) and (53), we deduce that $b_n \le b'_n \le M'(C)$. Hence we can conclude.

□

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